Recitation - Calculus II

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*Find the distance between the points $(3, 8, -1)$ and $(-2, 3, -6)$.

Solution. Let $(x_1, y_1, z_1) = (3, 8, -1)$ and $(x_2, y_2, z_2) = (-2, 3, -6)$.

$$\begin{aligned} distance &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-2-3)^2 + (3-8)^2 + (-6+1)^2} \\ &= \sqrt{75} = 5\sqrt{3} \end{aligned}$$

* Let $A = (-1, 2), B = (2, 0), C = (1, -3), D(0, 4)$. Find :

1. $\overrightarrow{AB} = (2 - (-1))\vec{i} + (0 - 2)\vec{j} = 3\vec{i} - 2\vec{j}$
2. $\overrightarrow{BA} = (-1 - 2)\vec{i} + (2 - 0)\vec{j} = -3\vec{i} + 2\vec{j}$
3. $\overrightarrow{AC} = (1 - (-1))\vec{i} + (-3 - 2)\vec{j} = 2\vec{i} - 5\vec{j}$
4. $\overrightarrow{BD} = (0 - 2)\vec{i} + (4 - 0)\vec{j} = -2\vec{i} + 4\vec{j}$

* Calculate the following for the vectors $\vec{u} = \vec{i} - \vec{j}$ and $\vec{v} = \vec{j} + 2\vec{k}$

1.
    - $\vec{u} + \vec{v} = (1+0)\vec{i} + (-1+1)\vec{j} + (2-0)\vec{k} = \vec{i} + 2\vec{k}$
    - $2\vec{u} - 3\vec{v} = (2-0)\vec{i} + (-2-3)\vec{j} + (0-6)\vec{k} = 2\vec{i} - 5\vec{j} - 6\vec{k}$
2.
    - $|\vec{u}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
    - $|\vec{v}| = \sqrt{1^2 + 2^2} = \sqrt{5}$
3. unit vectors $\hat{\vec{u}}$ and $\hat{\vec{v}}$

$$\hat{\vec{u}} = \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{i} - \vec{j}}{\sqrt{2}}, \quad \hat{\vec{v}} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{j} + 2\vec{k}}{\sqrt{5}}$$

4. $\vec{u} \bullet \vec{v} = 1 * 0 + (-1) * 1 + 0 * 2 = -1$
5. the angle (say, θ) between $\vec{u}$ and $\vec{v}$

$$\begin{aligned} \vec{u} \bullet \vec{v} &= |\vec{u}||\vec{v}| cos(\theta) \\ -1 &= \sqrt{2}\sqrt{5} cos(\theta) \\ \theta &= arccos\left(\frac{-1}{\sqrt{10}}\right) \end{aligned}$$

* Find the scalar projection of $\vec{u} = 3\vec{i} + 4\vec{j} - 5\vec{k}$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j} - 5\vec{k}$.

Solution.

$$\frac{\vec{u} \bullet \vec{v}}{|\vec{v}|} = \frac{9 - 16 + 25}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{18}{5\sqrt{2}}$$

* Use vectors to show that the triangle with vertices $(-1,1),(2,5)$ and $(10,-1)$ is right-angled.

Solution. Let $A(-1,1), B(2,5)$ and $C(10,-1)$. The right angle will be at one of the three vertices. For each vertices, we can investigate two vectors. If the angle between two vectors is 90^o then dot product will be zero since $\cos(90^o) = 0$.

$$\overrightarrow{AB} = 3\vec{i} + 4\vec{j}, \quad \overrightarrow{AC} = 11\vec{i} - 2\vec{j}, \quad \overrightarrow{BC} = 8\vec{i} - 6\vec{j}.$$

$$\overrightarrow{AB} \bullet \overrightarrow{AC} = 3 * 11 - 4 * 2 = 25$$
$$\overrightarrow{AB} \bullet \overrightarrow{BC} = 3 * 8 - 4 * 6 = 0$$
$$\overrightarrow{AC} \bullet \overrightarrow{BC} = 11 * 8 + 2 * 6 = 100$$

Since the dot product $\overrightarrow{AB} \bullet \overrightarrow{BC}$ is 0, the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is 90^o. Hence, the triangle is right-angled.

* Calculate $\vec{u} \times \vec{v}$ for $\vec{u} = \vec{j} + 2\vec{k}$ and $\vec{v} = -\vec{i} - \vec{j} + \vec{k}$.

Solution.

$$\begin{aligned}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \vec{k} \\ &= (1 - (-2))\vec{i} - (0 - (-2))\vec{j} + (0 - (-1))\vec{k} \\ &= 3\vec{i} - 2\vec{j} + \vec{k}\end{aligned}$$

* Find the cross product $\vec{a} \times \vec{b}$ for $\vec{a} = \vec{j} + 7\vec{k}, \; \vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$. Verify that $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Solution.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 7 \\ 2 & -1 & 4 \end{vmatrix} = 11\vec{i} + 14\vec{j} - 2\vec{k}$$

$$\vec{a} \bullet \left(\vec{a} \times \vec{b}\right) = 14 - 14 = 0 \Rightarrow \vec{a} \perp \left(\vec{a} \times \vec{b}\right)$$

$$\vec{b} \bullet \left(\vec{a} \times \vec{b}\right) = 22 - 14 - 8 = 0 \Rightarrow \vec{b} \perp \left(\vec{a} \times \vec{b}\right)$$

* Find two unit vectors perpendicular to the plane containing the points $(0,2,0),(1,0,0)$ and $(0,0,3)$. What is the area of the triangle with these vertices ?

Solution. Let $A(0,2,0), B(1,0,0)$ and $C(0,0,3)$.

$$\overrightarrow{AB} = \vec{i} - 2\vec{j}, \quad \overrightarrow{AC} = -2\vec{j} + 3\vec{k}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = -6\vec{i} - 3\vec{j} - 2\vec{k}, \quad |\vec{n}| = 7$$

Unit vectors are $\hat{\vec{n}} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{-6\vec{i} - 3\vec{j} - 2\vec{k}}{7}$ and -$\hat{\vec{n}} = \dfrac{6\vec{i} + 3\vec{j} + 2\vec{k}}{7}$

$$\text{Area} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|\vec{n}|}{2} = \frac{7}{2}$$