★ Determine the center, radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$.

Solution. $\sum_{n=1}^{\infty} \frac{\left(4\left(x+\frac{1}{4}\right)\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{4^n \left(x+\frac{1}{4}\right)^n}{n^3}$.

So, the center of convergence is $c = \frac{-1}{4}$.

By ratio test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^3} \frac{n^3}{(4x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{(4x+1)n^3}{(n+1)^3} \right| = |4x+1|$$

If $|4x+1| = 4|x-c| < 1$, then the series absolutely converges, so does converge. Meanwhile, $|x-c| < \frac{1}{4} = R$ where R is the radius of convergence. Then,

$$\left| x + \frac{1}{4} \right| < \frac{1}{4} \Rightarrow \frac{-1}{4} < x + \frac{1}{4} < \frac{1}{4} \Rightarrow \frac{-1}{2} < x < 0$$

<u>at $x = 0$</u> : $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by p-test ($p = 3 > 1$). So, the given series converges at $x = 0$.

<u>at $x = -\frac{1}{2}$:</u> $\sum_{n=1}^{\infty} \frac{\left(4(-\frac{1}{2})+1\right)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ by alternating series test,

- $a_n a_{n+1} = \frac{(-1)^n}{n^3} \frac{(-1)^{n+1}}{(n+1)^3} = \frac{(-1)^{2n+1}}{n^3(n+1)^3} < 0$
- $|a_{n+1}| = \frac{1}{(n+1)^3} < \frac{1}{n^3} = |a_n|$, since $(n+1)^3 > n^3$ for all $n = 1, 2, \ldots$
- $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^3} = 0$

the series converges at $x = -\frac{1}{2}$.

As a result, the interval of convergence $\left[-\frac{1}{2}, 0\right]$.

★ Find the parametric equations for the line through the point $(1,2,3)$ perpendicular to the plane $3x+2y+z=6$.

Solution.

$\vec{n}//\vec{v}$. Choose $\vec{v}=\vec{n}=3\vec{i}+2\vec{j}+\vec{k}$. Thus, the line equation in vector form :

$$\begin{aligned}\vec{r}&=\vec{r_0}+t\vec{v}\\&=(\vec{i}+2\vec{j}+3\vec{k})+t(3\vec{i}+2\vec{j}+\vec{k})\\&=(1+3t)\vec{i}+(2+2t)\vec{j}+(3+t)\vec{k}\end{aligned}$$

scalar parametric form :

$$\begin{aligned}x&=1+3t\\y&=2+2t\\z&=3+t\end{aligned}$$

standard form :

$$\frac{x-1}{3}=\frac{y-2}{2}=z-3$$

NOTE : To visualize figure while writing in computer is not easy. Therefore, I recommend you to draw the figures for these type of questions.

★ Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+2}{(n+1)^3}$ is convergent or divergent.

Solution. Let $a_n=\dfrac{n+2}{(n+1)^3}$. It is positive for all $n=1,2,\ldots$. So, we have a positive series. Let $b_n=\dfrac{n}{n^3}=\dfrac{1}{n^2}$. Then, the series $\sum_{n=1}^{\infty}b_n=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-test $(p=2>1)$, **AND**

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(n+2)n^2}{(n+1)^3}=\lim_{n\to\infty}\frac{n^3\left(1+\frac{2}{n}\right)}{n^3\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)}=1<\infty.$$

Therefore, $\sum_{n=1}^{\infty}a_n$ converges by limit comparison test.

★ Calculate $\dfrac{dz}{dz}$ given that $z = txy^2,\ x = t + \ln(y + t^2)$ and $y = e^t$.
Solution.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial t} \\ &= ty^2\left(1 + \frac{2t}{y+t^2}\right) + ty^2\frac{1}{y+t^2}e^t + 2yxte^t + xy^2\end{aligned}$$

★ Find the equation of the tangent plane $x^2y + yz^2 + xyz = 2$ at the point $(-1, 2, 1)$.

Solution. Let $P_0(x_0, y_0, z_0) = (-1, 2, 1)$. Then the normal vector of the tangent plane at the point P_0 is $\vec{n} = \vec{\nabla} f(P_0)$.

$$\begin{aligned}\vec{\nabla} f(x, y, x) &= f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k} \\ &= (2xy + yz)\vec{i} + (x^2 + z^2 + xz)\vec{j} + (2zy + xy)\vec{k} \\ \vec{\nabla} f(P_0) &= -2\vec{i} + \vec{j} + 2\vec{k} = \vec{n}\end{aligned}$$

The equation of the tangent plane : $\vec{n} \bullet \overrightarrow{PP_0} = 0$

$$\begin{aligned}n_1(x - x_0) + n_2(y - y0) + n_3(z - z_0) = 0, &\quad \text{where } \vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k} \\ -2(x + 1) + (y - 2) + 2(z - 1) = 0 & \\ -2x + y + 2z = 6 &\end{aligned}$$

★ Find and classify the critical points of the function

$$f(x,y) = 4xy - 2x^2 - y^4.$$

Solution.

$$f_1(x,y) = 4y - 4x = 0 \Rightarrow y = x$$
$$f_2(x,y) = 4x - 4y^3 = 0 \Rightarrow x = y^3$$

Putting $y = x$ into the second equation, we obtain

$$x = x^3 \Rightarrow x(1-x)(1+x) = 0 \Rightarrow x = 0,\ x = 1,\ x = -1$$
$$\Rightarrow y = 0,\ y = 1,\ y = -1$$

Thus, critical points are $(0,0), (1,1), (-1,-1)$.
To classify the critical points, we need

$$f_{11}(x,y) = -4,\ f_{12}(x,y) = f_{21}(x,y) = 4,\ f_{22}(x,y) = -12y^2.$$

<u>Classification of $(0,0)$</u>:

$$\left.\begin{array}{l} A = f_{11}(0,0) = -4 \\ B = f_{12}(0,0) = 4 \\ C = f_{22}(0,0) = 0 \end{array}\right\} \qquad B^2 - AC = 16 > 0, \quad (0,0) \text{ is a saddle point.}$$

<u>Classification of $(1,1)$</u>:

$$\left.\begin{array}{l} A = f_{11}(1,1) = -4 \\ B = f_{12}(1,1) = 4 \\ C = f_{22}(1,1) = -12 \end{array}\right\} \qquad B^2 - AC = 16 - 48 < 0 \text{ and } A = -4 < 0,$$

$(1,1)$ is local maximum of f.

<u>Classification of $(-1,-1)$</u>:

$$\left.\begin{array}{l} A = f_{11}(-1,-1) = -4 \\ B = f_{12}(-1,-1) = 4 \\ C = f_{22}(-1,-1) = -12 \end{array}\right\} \qquad B^2 - AC = 16 - 48 < 0 \text{ and } A = -4 < 0,$$

$(-1,-1)$ is also local maximum of f.