ON BLOW UP OF SOLUTIONS OF SECOND ORDER NONLINEAR PARABOLIC EQUATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

ABDULLAH ÖZBEKLER

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

JANUARY 1999

ABSTRACT

ON BLOW UP OF SOLUTIONS OF SECOND ORDER NONLINEAR PARABOLIC EQUATIONS

ÖZBEKLER, Abdullah
M.S., Department of Mathematics
Supervisor: Prof. Dr. Okay Çelebi

January 1999, 82 pages.

In this thesis, we shall give a survey of the initial and initial-boundary value problems for nonlinear parabolic differential equations. For the initial value problem we shall consider various extentions of an old result by H. Fujita for the initial value problem of the nonlinear heat equation

$$u_t = \Delta u + u^p \qquad x \in \mathbb{R}^N, \quad p > 1$$

with nonnegative initial values. The main results for the initial-boundary value problems, mentioned are due to C. Bandle and H. A. Levine. We will discuss the problems in different geometries.

Keywords: Nonlinear heat equation, Blow up, Critical exponents, Global ex-

istence, Global nonexistence

ÖZ

İKİNCİ DERECEDEN DOĞRUSAL OLMAYAN PARABOLİK DENKLEMLERİN ÇÖZÜMLERİNİN PATLAMASI

ÖZBEKLER, Abdullah

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi: Prof. Dr. A. Okay ÇELEBİ

Ocak 1999, 82 sayfa.

Bu tezde, doğrusal olmayan parabolik diferensiyel denklemler için başlangıç ve sınır değer ile başlangıç değer problemlerinin bir araştırmasını vereceğiz. Başlangıç değer problemi için, H. Fujita'nın doğrusal olmayan

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N \quad p > 1$$

ısı denklemi için negatif olmayan başlangıç değerli problemle ilgili sonuçlarının çeşitli genişletmelerini çalışacağız. Burada başlangıç ve sınır değer problemleri için bahsetmek istediğimiz başlıca sonuçlar, C. Bandle ve H. A. Levin'in sonuçlarıdır. Problemleri farklı geometrilerde tartışacağız.

Anahtar Sözcükler: Doğrusal olmayan ısı denklemi, Patlama, Kritik kuvvet,

Global varlık, Global yokluk.

ACKNOWLEDGEMENTS

I would like to express my deepest thanks to my supervisor Prof. Dr. A. Okay ÇELEBİ and Prof. Dr. Varga KALANTAROV for their motivation, guidance, encouragement and insight during the preparation of this thesis.

I also thank to all my friends, especially Ali Akçay, Davut Uğurlu, Erdal Karapınar, Fedai Ahi, Mustafa Arslan, Mustafa Polat and Müjdat Kaya for their moral support and help specifically in the exposition of this thesis.

Finally, I would like to dedicate this thesis to my family...

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This thesis is a survey on the recent literature on the role of the size of nonlinearity for the occurance of blow up. As an example of the type of results we may mention the clasical result of Fujita [4]: He considered the initial value problem

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N, \quad t > 0, \quad p > 1 \tag{1.1}$$

$$u(x,0) = a(x), \qquad x \in \mathbb{R}^N \tag{1.2}$$

where Δ denotes the N-dimensional Laplace operator) and interested in non-negative solutions which, for fixed t, decay at infinity. He proved the following result where the critical exponent $pc(N) := 1 + 2/N$,

A. If $1 < p < pc(N)$, then the only nonnegative global (in time) solution of (1.1)-(1.2) is $u \equiv 0$.

B. If $p > pc(N)$, then there exist global positive solutions of (1.1)-(1.2) for sufficiently small initial values.

Several comments about this result are in order. First, when the solution fails to exist globally, it actually does blow up pointwise. This was proven by Bandle and Levine [3].

Levine, Lieberman and Meier [9] considered the problem (Generalized Mean Curvature):

$$u_t = \Phi u + u^p \qquad (x,t) \in \mathbb{R}^N \times (0,T) \tag{1.3}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^N \tag{1.4}$$

where

$$\Phi u = \text{div}\{\psi[(1+|\nabla u|^2)^{1/2}]\nabla u\}$$

with

(i) $\psi \in C^{1,\alpha}([1,\infty))$; $\qquad \psi(1) = \psi_0 > 0$,

(ii) $0 \leq \psi'(s) + \psi(s) \leq (1+\theta)\psi(s) \qquad (\theta > 0)$,

(iii) $\psi(s) \leq \psi_M, \qquad \psi_M < \infty$.

They have obtained the interesting result that the critical exponent for the problem (1.3)-(1.4) is $pc(N)$, i.e.;

A. If $1 < p < pc(N)$, then there are no nontrivial positive solutions of (1.3)-(1.4) .

B. If $p > pc(N)$, then there exist both positive global solutions of (1.3)-(1.4) and solutions of (1.3) -(1.4) which blow up in finite time, but no claim is made when $p = pc(N)$.

Instead of (1.1)-(1.2) , the following initial boundary value problem was considered in [3], [10]:

$$\begin{aligned} u_t &= \Delta u + u^p, && (x,t) \in D \times (0,T) && (1.5)\\ u(x,0) &= u_0(x), && x \in D && (1.6)\\ u(x,t) &= 0, && (x,t) \in \partial D \times [0,T) && (1.7) \end{aligned}$$

where

$$D = \{(r,\theta) | r > 0, \theta \in \Omega \subset S^{N-1}\}.$$

The next result shows that the critical exponent for cones:

$$pc(D) = 1 + \frac{2}{2 - \gamma_-}$$

where γ_- is the root of the quadratic

$$x(N-2+x)=\omega_1$$

(Here ω_1 denotes the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator on Ω). Bandle and Levine [3] proved that if $1<p\leq pc(D)$, then (1.5)-(1.7) has no nontrivial, global, positive solutions. Later Levine and Meier [10] showed that $pc(D)$ is indeed the critical exponent, in other words if $p>pc(D)$ then global, positive, small data solutions do exist.

Bandle and Levine [3] showed also that if $1<p<1-2/\gamma_-$, then there are no stationary solutions of (1.5)-(1.7) , that is, there exits no $w(r,\theta)>0$ such that

$$\Delta w+w^p=0 \quad \text{in} \quad D, \qquad\qquad w(r,\theta)=0 \quad \text{on} \quad \partial D$$

where D is the cone in $\mathbb{R}^N$ with vertex at origin.

Meier [12] considered for positive constants c, b, the following initial boundary value problem:

$$u_t = \Delta u+ce^{bt}u^p, \qquad x\in D, \quad t>0 \tag{1.8}$$

$$u(x,t) = 0, \qquad x\in\partial D, \quad t>0 \tag{1.9}$$

$$u(x,0) = u_0(x)\geq 0 \qquad x\in D \tag{1.10}$$

where D is a bounded domain in $\mathbb{R}^N$ with sectionaly smooth boundary, u_0 is bounded and $p>1$. He proved that:

A. If $1<p<1+b/\lambda_1$, then (1.8)-(1.10) has no nontrivial, positive, global solutions.

B. If $p > 1 + b/\lambda_1$, then (1.8)-(1.10) has both nontrivial, global small data solutions and solutions which blow up in a finite time.
So the critical exponent for (1.8)-(1.10) is

$$pc(BD) = 1 + \frac{b}{\lambda_1}.$$

Here λ_1 is the first eigenvalue of the Dirihlet problem for the Laplace operator in D

In1994, Bandle and Levine [3] studied the global existence of nonnegative solutions of the Cauchy problem

$$u_t - \Delta u = u^p + (\vec{b}, \nabla u), \qquad (x,t) \in D \times (0,T) \tag{1.11}$$

$$u(x,0) = u_0(x) \geq 0, \qquad x \in D \tag{1.12}$$

where D is either $\mathbb{R}^N$ or a cone in $\mathbb{R}^N$, $\vec{b} = \vec{b}(x,u)$ and $p > 1$. Their aim was to extend H. Fujita's result which corresponds to the case $\vec{b} \equiv 0$. If, in particular $\vec{b} = \vec{b}(x)$, div$\vec{b} = 0$ and $|x||\vec{b}(x)|$ is bounded, the Fujita-like theorem is recovered as in [4],[11]. If the domain D is a shift-invariant cone then they proved a result similar to the one given for (1.5)-(1.7) in [3]. Finally they showed that Kaplan's convexity method is applicable to nonlinear convection of the type $(\vec{b}(u), \nabla u) = \text{div}\vec{B}(u)$, $D = \mathbb{R}^N$. Under suitable growth conditions imposed on $|\vec{B}(u)|$, they proved that all solutions of (1.11)-(1.12) must blow up in finite time. This case was also studied by Aguare and Escobedo [1] in 1993.

In 1996, Galaktionov and Levine [5] considered nonnegtive solutions of initial boundary value problems for parabolic equations $u_t = u_{xx}$, $u_t = (u^m)_{xx}$

and $u_t = (|u_x|^{m-1}u_x)_x$ $(m > 1)$ for $x > 0$, $t > 0$ with nonlinear boundary conditions $-u_x = u^p$, $-(u^m)_x = u^p$ and $-|u_x|^{m-1}u_x = u^p$ for $x = 0$, $t > 0$ where $p > 0$ respectively. The initial function was assumed to be bounded, smooth and to have, in the latter two cases, compact support. They proved that for each problem there exit positive critical values p_0, p_c (with $p_0 < p_c$) such that for $p \in (0, p_0]$, all solutions are global while for $p \in (p_0, p_c]$ any solution $u \not\equiv 0$ blows up in a finite time and for $p > p_c$ small data solutions exit globally in time while large data solutions are nonglobal. They obtained the critical exponents $p_c = 2$, $p_c = m+1$ and $p_c = 2m$, and $p_0 = 1$, $p_0 = (m+1)/2$ and $p_0 = 2m/(m+1)$ respectively.

Recently, C. S. Jiang and C. H. Xie [7] considered the blow up problem for the initial boundary value problem for quasilinear parabolic equation

$$\psi'(u)u_t - \delta u = h(x,t)F(u), \qquad x \in D, \quad t > 0 \tag{1.13}$$

$$u(x,0) = u_0(x), \qquad x \in \bar{D} \tag{1.14}$$

$$u(x,t) = 0, \qquad x \in \partial D, \quad t > 0. \tag{1.15}$$

Here $D \subset \mathbb{R}^N$ is a bounded or unbounded domain. With the help of Green's function, they obtained conditions for existence of global solutions and blow up of local solution to problem (1.13)-(1.15).

Recently, Y. Qi [14] studied the Cauchy problem in $\mathbb{R}^n$ for general parabolic equations which take the form

$$u_t = \Delta u^m + t^s|x|^\sigma u^p \tag{1.16}$$

with nonnegative initial value. Here $s \geq 0$, $m > (n-2)/n$, $p > \max(1, m)$ and $\sigma > -1$ if $n = 1$ or $\sigma > -2$ if $n \geq 2$. He proved among other things, that for $p \leq p_c$, where $p_c \equiv m + s(m-1) + (2+2s+\sigma)/n > 1$, every nontrivial solution of (1.16) blows up in finite time. But for $p > p_c$ there exits a positive global solution of (1.16).

In this thesis we will make use of the following definitions

Definition 1.0.1 *A non-negative function $u = u(t,x)$ is called a regular solution of (1.1)-(1.2) in [0,T], T being a positive number, if u, $\nabla_x u$, $\nabla_x \nabla_x u$ and u_t all exist and are continuous in $Q_T = [0,T] \times \mathbb{R}^m$ and if (1.1)-(1.2) is satisfied. A regular solution u of (1.1)-(1.2) in $[0,\infty]$ is a function whose restriction to $[0,T] \times \mathbb{R}^m$ is a regular solution of (1.1)-(1.2) in [0,T] for any $T > 0$.*

Definition 1.0.2 *Let T be a positive number. $\mathcal{E}$[0,T] is the set of all continuous functions $u = u(t,x)$ defined in $[0,T] \times \mathbb{R}^m$ satisfying*

$$|u(t,x)| \leq Mexp(|x|^\beta) \qquad (0 \leq t \leq T, x \in \mathbb{R}^m)$$

with some constants M and β subject to $M > 0$ and $0 < \beta < 2$. M and β may depend on u. Furthermore, $\mathcal{E}[0,\infty)$ is the set of all u whose restriction to $[0,T] \times \mathbb{R}^m$ belongs to $\mathcal{E}[0,T]$ for any $T > 0$.

We note that $u^{1+\alpha} \in \mathcal{E}[0,T]$ if $u \in \mathcal{E}[0,T]$ and $u \geq 0$.

Definition 1.0.3 *If u is a regular solution of (1.1)-(1.2) in $[0,T]$ and at the same time $u \in \mathcal{E}[0,T]$, then u is called a regular solution of (1.1)-(1.2) in $\mathcal{E}[0,T]$. Here we may replace $\mathcal{E}[0,T]$ by $\mathcal{E}[0,\infty)$. A regular solution u of (1.1)-(1.2) in $\mathcal{E}[0,\infty)$ is also called a global solution of (1.1) -(1.2) in $\mathcal{E}[0,\infty)$.*

Next we specify the class of initial values. We assume that the initial value $a = a(x)$ of (1.1)-(1.2) is taken from the class $\mathcal{A}$ described in

Definition 1.0.4 $\mathcal{A}$ *is the set of all non-negative functions* $a = a(x)$ *on* $\mathbb{R}^m$ *such that* $a, \nabla_x a$ *and* $\nabla_x \nabla_x a$ *are all continuous and bounded there.*

Definition 1.0.5 $\mathcal{L}[0,\infty)$ *is the set of all non-negative continuous functions* $u = u(t,x)$ *defined in* $[0,\infty) \times \mathbb{R}^m$ *such that the inequality*

$$0 \leq u(t,x) \leq MH(t+\gamma, x), \qquad (t \geq 0, x \in \mathbb{R}^m),$$

is satisfied for some constant M *which may depend on* $u(t,x)$ *and* $H(t,x)$ *is the Green's function for the heat equation.*

Definition 1.0.6 *[13] (Upper Solution)* $\bar{u}$ *is the upper solution (supersolution) of (1.1) -(1.2) if*

$$\bar{u}_t \geq \Delta \bar{u} + \bar{u}^p, \qquad x \in \mathbb{R}^N, \quad t > 0 \tag{1.17}$$

$$\bar{u}(x,0) \geq a(x), \qquad x \in \mathbb{R}^N. \tag{1.18}$$

Definition 1.0.7 *[13] (Lower Solution)* $\underline{u}$ *is the lover solution (subsolution) of (1.1) -(1.2) if*

$$\underline{u}_t \leq \Delta \underline{u} + \underline{u}^p, \qquad x \in \mathbb{R}^N, \quad t > 0 \tag{1.19}$$

$$\underline{u}(x,0) \leq a(x), \qquad x \in \mathbb{R}^N. \tag{1.20}$$

Jensen's Inequality, [6]: Suppose that $\alpha \leq f(x) \leq \beta$, where α and β may be finite or infinite, and that $f(x)$ is almost always different from α and β;

that $p(x)$ is a weight function; and that $\phi''(t)$ is positive (i. e. convex function) and finite for $\alpha < t < \beta$. Then

$$\phi\Big(\frac{\int f(x)p(x)dx}{\int p(x)dx}\Big) \leq \frac{\int \phi(f(x))p(x)dx}{\int p(x)dx},$$

whenever the right-hand side exists and is finite; and there is equality only when $f(x) \equiv C$.

CHAPTER 2

BLOWING UP OF SOLUTIONS FOR A CAUCHY PROBLEM

2.1 Introduction

In this chapter we will discuss the properties of the solutions of the problem:

$$u_t = \Delta u + u^{1+\alpha} \qquad x \in \mathbb{R}^m, \quad t > 0 \tag{2.1}$$

$$u(x,0) = a(x) \qquad x \in \mathbb{R}^m \tag{2.2}$$

where $a(x) \in \mathcal{A}$. Let us describe the results in a rough way. If $0 < m\alpha < 2$, then every non-negative solution of (2.1) -(2.2) blows up eventually except the trivial slution $u \equiv 0$. If $2 < m\alpha$, there are many non-negative initial values $a = a(x)$ which give global solutions. This appears somewhat remarkable in asmuch as the inevitable blowing up occurs rather in the case of smaller α. [4]

2.2 Global Nonexistence of Solutions

Now we are ready to prove our main results. As a preparation, we some propositions. [4]

Proposition 2.2.1 *Let $u(x,t)$ be a regular solution of (2.1) -(2.2) in $\mathcal{E}[0,T]$ for $T > 0$. Then $u(x,t)$ satisfies the integral equation $u = u_0 + \phi u$ in $0 \le t \le T$, where*

$$u_0(x,t) = \int_{R^m} H(t, x-y) a(y) dy$$

$$\phi u(x,t) \;=\; \int_0^t ds \int_{R^m} H(t-s, x-y) u^{1+\alpha}(s,y) dy.$$

Proof. Take $\rho \in C_0^\infty(\mathbb{R}^m)$ such that $0 \le \rho \le 1$ and

$$\rho(x) := \begin{cases} 1 & , |x| \le 1 \\ 0 & , |x| \ge 2 \end{cases}. \tag{2.3}$$

Define $\rho_N(x) := \rho(x/N)$ (called truncating functions). We put $\nu_N(x,t) = \rho_N(x)u(x,t)$ $(N = 1, 2, ...)$. Then we have

$$\frac{\partial \nu_N}{\partial t} = \Delta \nu_N + \rho_N u^{1+\alpha} - 2\nabla \rho_N . \nabla u - \Delta \rho_N u$$

and $\nu_N(0,x) = \rho_N(x)u(x,0) = \rho_N(x)a(x)$.

$$\begin{aligned}
\frac{\partial \nu_N}{\partial t} &= \rho_N u_t \\
&= \rho_N \{\Delta u + u^{1+\alpha}\} \\
&= \Delta \nu_N - \Delta \nu_N + \rho_N \Delta u + \rho_N u^{1+\alpha} \\
&= \Delta \nu_N - \Delta(\rho_N u) + \rho_N \Delta u + \rho_N u^{1+\alpha} \\
&= \Delta \nu_N - \rho_N \Delta u - 2\nabla \rho_N . \nabla u - u \Delta \rho_N + \rho_N \Delta u + \rho_N u^{1+\alpha} \\
&= \Delta \nu_N + \rho_N u^{1+\alpha} - 2\nabla \rho_N . \nabla u - \Delta \rho_N u.
\end{aligned}$$

If we multiply both sides, by $H(t-s, x-y)$ and integrate over $\mathbb{R}^m$ and on $[0,t]$, we get

$$\begin{aligned}
& \int_0^t ds \int_{R^m} \frac{\partial \nu_N}{\partial s}(s,y) H(t-s, x-y) dy \\
= \;& \int_0^t ds \int_{R^m} \Delta_y \nu_N(s,y) H(t-s, x-y) dy
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t ds \int_{R^m} H(t-s,x-y)\rho_N(y).u^{1+\alpha}(s,y)dy \\
&- 2\int_0^t ds \int_{R^m} H(t-s,x-y)\nabla_y\rho_N(y).\nabla_y u(s,y)dy \\
&- \int_0^t ds \int_{R^m} H(t-s,x-y)\Delta_y\rho_N(y)u(s,y)dy \\
&= \int_0^t ds \int_{R^m} \Delta_y\nu_N(s,y)H(t-s,x-y)dy + V_2 - 2V_3 - V_4
\end{aligned}
$$

where

$$
\begin{aligned}
V_2 &= \int_0^t \int_{R^m} H(t-s,x-y)\rho_N(y)u(s,y)^{1+\alpha}dy, \\
V_3 &= \int_0^t \int_{R^m} H(t-s,x-y)\nabla_y\rho_N(y)\nabla_y u(s,y)dy, \\
V_4 &= \int_0^t \int_{R^m} H(t-s,x-y)\Delta_y\rho_N(y)u(s,y)dy.
\end{aligned}
$$

If we apply Green's identity, we get

$$
\begin{aligned}
& \int_0^t ds \int_{R^m} \frac{\partial\nu_N}{\partial s}(s,y)H(t-s,x-y)dy \\
=& \int_0^t ds \int_{R^m} \nu_N(s,y)\Delta_y H(t-s,x-y)dy \\
+& \int_0^t ds \int_{\partial R^m} \left\{ H(t-s,x-y)\frac{\partial\nu_N}{\partial n}(s,y) + \nu_N(s,y)\frac{\partial H}{\partial n}(t-s,x-y) \right\} dS \\
+& V_2 - 2V_3 - V_4.
\end{aligned}
$$

Since $\nu_N(s,y) = 0$ and $H(t-s,x-y) = 0$ on boundary $\mathbb{R}^m$, we have

$$
\begin{aligned}
&= \int_0^t ds \int_{R^m} \nu_N(s,y)\Delta_y H(t-s,x-y)dy + V_2 - 2V_3 - V_4 \\
&= \int_0^t ds \int_{R^m} \nu_N(s,y)\Delta_x H(t-s,x-y)dy + V_2 - 2V_3 - V_4 \\
&= \int_0^t ds \int_{R^m} \nu_N(s,y)\frac{\partial}{\partial t} H(t-s,x-y)dy + V_2 - 2V_3 - V_4 \\
&= -\int_0^t ds \int_{R^m} \nu_N(s,y)\frac{\partial}{\partial s} H(t-s,x-y)dy + V_2 - 2V_3 - V_4
\end{aligned}
$$

then we have

$$\begin{aligned}
V_2 - 2V_3 - V_4 &= \int_0^t ds \int_{R^m} \Big\{ \frac{\partial}{\partial s}\nu_N(s,y)H(t-s,x-y) \\
&+ \nu_N(s,y)\frac{\partial H}{\partial s}(t-s,x-y)\Big\} dy \\
&= \int_0^t \frac{\partial}{\partial s}\Big\{(4\pi(t-s))^{-m/2}\int_{R^m} \nu_N(s,y)exp(-\frac{|x-y|^2}{4(t-s)})dy\Big\} ds
\end{aligned}$$

if we let $\xi = \frac{y-x}{2\sqrt{t-s}}$ and $dy = (4(t-s))^{m/2}d\xi$ then we get

$$\begin{aligned}
V_2 - 2V_3 - V_4 &= \pi^{-m/2}\int_0^t \frac{\partial}{\partial s}\Big\{\int_{R^m} \nu_N(s, x+2\sqrt{t-s}\xi)exp(-|\xi|^2)d\xi\Big\} ds \\
&= \pi^{-m/2}\nu_N(s,x)\int_{R^m} exp(-|\xi|^2)d\xi \\
&- \pi^{-m/2}\int_{R^m} \nu_N(0, x+2\sqrt{t}\xi)exp(-|\xi|^2)d\xi \\
&= \nu_N(s,x) - \pi^{-m/2}\int_{R^m} \nu_N(0,x+2\sqrt{t}\xi)exp(-|\xi|^2)d\xi
\end{aligned}$$

Substituting $x + 2\sqrt{t}\xi = y$ we have

$$\begin{aligned}
V_2 - 2V_3 - V_4 &= \nu_N(s,x) - (4\pi t)^{-m/2}\int_{R^m} \nu_N(0,y)exp(-\frac{|x-y|^2}{4t})dy \\
&= \nu_N(s,x) - \int_{R^m} H(t,x-y)\rho_N(y)a(y)dy \\
&= \nu_N(t,x) - V_1.
\end{aligned}$$

where

$$V_1 = \int_{R^m} H(t,x-y)\rho_N(y)a(y)dy.$$

Consequently we get

$$\nu_N(t,x) = V_1 + V_2 - 2V_3 - V_4. \tag{2.4}$$

Now an easy calculation gives that

$$\begin{aligned}
|V_1 - \int_{R^m} H(t, x-y)a(y)dy| &= |\int_{R^m} H(t, x-y)\rho_N(y)a(y)dy \\
&- \int_{R^m} H(t, x-y)a(y)dy| \\
&\le \int_{R^m} H(t, x-y)|\rho_N(y) - 1|a(y)dy
\end{aligned}$$

which yields

$$V_1 \to \int_{R^m} H(t, x-y)a(y)dy \quad \text{as} \quad N \to \infty.$$

We know that $u(t,x) \in \mathcal{E}[0,T]$. So

$$|u^{1+\alpha}(t,x)| \le Mexp(|x|^\beta), \qquad (0 \le s \le T, y \in \mathbb{R}^m).$$

Thus we have

$$\begin{aligned}
|V_2(t,x) - \phi u(t,x)| &\le \int_0^t ds \int_{R^m} H(t-s, x-y)\rho_N(y)u^{1+\alpha}(s,y)dy \\
&- \int_0^t ds \int_{R^m} H(t-s, x-y)u^{1+\alpha}(s,y)dy \\
&\le \int_0^t ds \int_{R^m} H(t-s, x-y)|\rho_N(y) - 1|u^{1+\alpha}(s,y)dy, \\
&= \int_0^t ds \int_{|y|\ge N} H(t-s, x-y)|\rho_N(y) - 1|u^{1+\alpha}(s,y)dy \\
&+ \int_0^t ds \int_{|y|\le N} H(t-s, x-y)|\rho_N(y) - 1|u^{1+\alpha}(s,y)dy \\
&= \int_0^t ds \int_{|y|\ge N} H(t-s, x-y)|\rho_N(y) - 1|u^{1+\alpha}(s,y)dy,
\end{aligned}$$

Therefore

$$\begin{aligned}
|V_2(t,x) - (\phi u)(t,x)| &\le \int_0^t ds \int_{|y|\ge N} H(t-s, x-y)u^{1+\alpha}(s,y)dy \\
&\le M \int_0^t ds \int_{|y|\ge N} H(t-s, x-y)exp(|y|^\beta)dy \\
&= M \int_0^t \varphi_N(s)ds,
\end{aligned}$$

where

$$\varphi_N(s) = \int_{|y|\geq N} H(t-s, x-y) exp(|y|^\beta) dy.$$

For each x and t

$$\begin{aligned}
0 \leq \varphi_N(s) &\leq \int_{R^m} H(t-s, x-y) exp(|y|^\beta) dy \\
&\leq C \int_{R^m} exp(-|\xi|^2) exp(|x + 2\sqrt{t-s}\xi|^\beta) d\xi \\
&\leq C \int_{R^m} exp(-|\xi|^2) exp((|x| + 2\sqrt{t}|\xi|)^\beta) d\xi \\
&\leq C' \int_{R^m} exp(-|\xi|^2) exp(2^\beta t^{\beta/2} |\xi|^\beta) d\xi
\end{aligned}$$

where C' is a constant depending on $|x|$. Since the last integral is bounded, $\varphi_N(s)$ is bounded unifomly by a constant.

Now claim that $\varphi_N(s) \to 0$ as $N \to \infty$. If we let $y = x + 2\sqrt{t-s}\xi$ then we get

$$\begin{aligned}
\varphi_N(s) &= \int_{|y|\geq N} H(t-s, x-y) exp(|y|^\beta) dy \\
&= C \int_{|\xi|\geq M} exp(-|\xi|^2) exp(|x + 2\sqrt{t-s}\xi|^\beta) d\xi \\
&\leq C' \int_{|\xi|\geq M} exp(-|\xi|^2) exp(C_0 |\xi|^\beta) d\xi
\end{aligned}$$

where $M = \frac{N-|x|}{2\sqrt{t-s}}$, $C_0 = 2^\beta t^{\beta/2}$ and C' is a constant depending on $|x|$. So we get

$$\begin{aligned}
\varphi_N(s) &= C' \int_{|\xi|\geq M} exp(-|\xi|^2 + C_0|\xi|^\beta) d\xi \\
&= C' \prod_{i=1}^{m} \int_{|\xi_i|\geq M} exp(-|\xi_i|^2 + C_0 |\xi_i|^\beta) d\xi_i \\
&= C' 2^m \prod_{i=1}^{m} \int_M^\infty exp(-\xi_i^2 + C_0 \xi_i^\beta) d\xi_i
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C'' \prod_{i=1}^{m} \int_M^\infty \frac{d\xi_i}{\xi_i^2} \\
&= \; C'' \prod_{i=1}^{m} \frac{1}{M} \\
&= \; \frac{C''}{M^m} \to 0 \;\text{ as }\; M \to \infty \;\;\text{(i.e. } N \to \infty).
\end{aligned}
$$

Therefore we can apply the Lebesque convergence theorem [15], to find

$$
|V_2(t,x) - (\phi u)(t,x)| \leq M \int_0^t \varphi_N(s)ds \quad \to 0, \qquad (N \to \infty).
$$

We now claim $V_3 \to 0$ and $V_4 \to 0$ is dealt with easily by means of $|\Delta \rho_N(y)| \leq CN^{-2}$. First let us with V_4.

$$
\begin{aligned}
|V_4| \;&\leq\; \int_0^t ds \int_{R^m} H(t-s,x-y)|u(s,y)||\Delta\rho_N(y)|dy \\
&\leq\; CN^{-2} \int_0^t ds \int_{R^m} exp(-|\xi|^2) exp((|x+2\sqrt{t-s}\xi|)^\beta)d\xi \\
&\leq\; C'N^{-2} \int_0^t ds \int_{R^m} exp(-|\xi|^2) exp(2^\beta t^\beta |\xi|^\beta)d\xi \\
&\leq\; C''N^{-2} \to 0 \;\text{ as }\; N \to \infty.
\end{aligned}
$$

where C', C'' are constants depending on t and $|x|$.

Now consider

$$
V_3 = \int_0^t ds \int_{R^m} \Big\{ H(t-s,x-y)\nabla\rho_N(y) \Big\} \nabla u(s,y)dy
$$

Using integration by parts, we get

$$
V_3 = -\int_0^t ds \int_{R^m} \nabla \Big\{ H(t-s,x-y)\nabla\rho_N(y) \Big\} u(s,y)dy,
$$

since $H(t-s,x-y)\nabla\rho_N(y)u(s,y) \mid_{\partial R^m} = 0$. Then

$$
\begin{aligned}
V_3 = -\int_0^t ds \int_{R^m} \Big\{ &\nabla H(t-s,x-y)\nabla\rho_N(y) \\
&+ H(t-s,x-y)\delta\rho_N(y) \Big\} u(s,y)dy, \\
= -\tilde{V}_3 - V_4&,
\end{aligned}
$$

where

$$\tilde{V}_3 = \int_0^t ds \int_{R^m} \nabla H(t-s, x-y).\nabla \rho_N(y).u(s,y)dy.$$

In order to estimate $|\tilde{V}_3|$, we note $|\nabla \rho_N(y)| \leq CN^{-1}$. Now compute

$$|\nabla_x H(t,x)|.$$

$$\begin{aligned} |\nabla_x H(t,x)|^2 &= \sum_{i=1}^{m} |\frac{\partial}{\partial x_i} H(t,x)|^2 \\ &= C' t^{-m} exp(-\frac{|x|^2}{2t}) t^{-2}, \end{aligned}$$

where C' is a constant depending on $|x|$. Then

$$\begin{aligned} |\nabla_x H(t,x)| &= C' t^{-m/2} exp(-\frac{|x|^2}{4t}) t^{-1}, \\ &\leq C' t^{-m/2} exp(-\frac{|x|^2}{9t}) t^{-1/2}, \\ &= C' t^{-(m+1)/2} exp(-\frac{|x|^2}{9t}). \end{aligned}$$

Therefore we have

$$|\nabla_x H(t,x)| = C' t^{-(m+1)/2} exp(-\frac{|x|^2}{9t}). \tag{2.5}$$

We may suppose that

$$|u(s,y)| \leq M exp(|y|^\beta), \qquad (0 \leq s \leq T, y \in R^m) \tag{2.6}$$

with some constants $M > 0$ and $\beta \in (0,2)$, $u(s,y) \in \mathcal{E}[0,T]$. Then

$$\begin{aligned} |\tilde{V}_3| &= |\int_0^t ds \int_{R^m} \nabla H(t-s, x-y).\nabla \rho_N(y).u(s,y)dy| \\ &\leq CN^{-1} \int_0^t t^{-(m+1)/2} ds \int_{R^m} exp(-\frac{|x-y|^2}{9(t-s)}) exp(|y|^\beta) dy, \end{aligned}$$

if we let $y = x + 3\sqrt{t-s}\xi$, then

$$\begin{aligned} |\tilde{V}_3| &\le CN^{-1}\int_0^t (t-s)^{-1/2}ds \int_{R^m} exp(-|\xi|^2)exp(|x+3\sqrt{t-s}\xi|^\beta)d\xi \\ &\le C'N^{-1}\int_0^t (t-s)^{-1/2}ds \int_{R^m} exp(-|\xi|^2)exp(3^\beta(t-s)^{\beta/2}|\xi|^\beta)d\xi \\ &\le C''N^{-1} \to 0 \ \text{ as } \ N \to \infty, \end{aligned}$$

with C'' depending only on x and t. Thus we have $|\tilde{V}_3| \to 0$ as $N \to \infty$ and consequently

$$|V_3| \le |\tilde{V}_3| + |V_4| \to 0 \ \text{ as } \ N \to \infty.$$

Since

$$\lim_{N\to\infty} \nu_N(x,t) = \lim_{N\to\infty} \rho_N(x)u(x,t) = u(x,t),$$

from (2.4) we get

$$u(x,t) = u_0(x,t) + \phi u(x,t). \tag{2.7}$$

Remark. Let $u(x,t)$ be as in the preceding proposition. In addition, assume that the initial value $a(x)$ of $u(x,t)$ is not trivial. Then $u = u(x,t) > 0$ if $t > 0$.

Proposition 2.2.2 *Let $u(x,t)$ be a regular solution of (2.1) -(2.2) in $\mathcal{E}[0,T]$. Then we have*

$$\frac{\partial}{\partial x_j}u(x,t) \in \mathcal{E}[0,T], \qquad (j = 1,2,...,m).$$

Proof. Suppose that $\nu(y,s)$ is a function in $\mathcal{E}[0,T]$ satisfying (2.6). Put

$$w(x,t) = \int_0^t ds \int_{R^m} H(t-s, x-y)\nu(y,s)dy. \tag{2.8}$$

Then

$$\frac{\partial}{\partial x_j}w(x,t)=\int_0^t ds\int_{R^m}\frac{\partial}{\partial x_j}H(t-s,x-y)\nu(y,s)dy,$$

by means of (2.5) we have

$$|\frac{\partial}{\partial x_j}H(t-s,x-y)|\le Ct^{-(m+1)/2}exp(-\frac{|x|^2}{9t}).$$

For any positive constant λ we can choose $N>0$ and γ in $\beta<\gamma<2$ satisfying

$$exp(\lambda|x|^{\beta})\le Nexp(|x|^{\beta}),\qquad (x\in\mathbb{R}^m),$$

we see that

$$\begin{aligned}|\frac{\partial w}{\partial x_j}(x,t)| &\le \int_0^t ds\int_{R^m}|\frac{\partial}{\partial x_j}H(t-s,x-y)||\nu(y,s)|dy\\ &\le C\int_0^t(t-s)^{-(m+1)/2}ds\int_{R^m}exp(-\frac{|x-y|^2}{9(t-s)})exp(|y|^{\beta})dy,\end{aligned}$$

if we let $y=x+3\sqrt{t-s}\xi$ then

$$\begin{aligned}|\frac{\partial w}{\partial x_j}(x,t)| &\le C\int_0^t(t-s)^{-1/2}ds\int_{R^m}exp(-|\xi|^2)exp(|x+3\sqrt{t-s}\xi|^{\beta})d\xi,\\ &\le C\int_0^t(t-s)^{-1/2}ds\int_{R^m}exp(-|\xi|^2)exp((|x|+3\sqrt{t}|\xi|)^{\beta})d\xi,\\ &\le Cexp(|x|^{\gamma})\int_0^t(t-s)^{-1/2}ds\int_{R^m}exp(-|\xi|^2)exp(3^{\beta}t^{\beta/2}|\xi|^{\beta})d\xi,\\ &\le C'exp(|x|^{\gamma})\int_0^t(t-s)^{-1/2}ds,\\ &\le C''exp(|x|^{\gamma}),\end{aligned}$$

where C'' is a constant depending on t and γ in $\beta<\gamma<2$. This shows

$$\frac{\partial}{\partial x_j}w(x,t)\in\mathcal{E}[0,T],\qquad (j=1,2,...,m).$$

Since $a(y) \in \mathcal{A}$ then

$$
\begin{aligned}
|\frac{\partial}{\partial x_j} u_0(x,t)| &\leq |\nabla_x u_0(x,t)| \\
&= |\int_{R^m} \nabla_x H(t, x-y) a(y) dy| \\
&\leq C \int_{R^m} |\nabla_x H(t, x-y)| dy \\
&\leq C' t^{-(m+1)/2} \int_{R^m} exp(-\frac{|x-y|^2}{9t}) dy, \\
&\leq C'' t^{-1/2} \pi^{m/2} = C_1
\end{aligned}
$$

where C_1 is a constant depending on t. On the other hand we can put $\nu(x,t) = u^{1+\alpha}(x,t)$ in (2.8) since $u \in \mathcal{E}[0,T]$ implies that $u^{1+\alpha} \in \mathcal{E}[0,T]$. Hence we get

$$
\frac{\partial}{\partial x_j}(\phi u)(x,t) \in \mathcal{E}[0,T].
$$

Since $u = u_0 + \phi u$, we have

$$
|\frac{\partial}{\partial x_j} u(x,t)| \leq |\frac{\partial}{\partial x_j} u_0(x,t)| + |\frac{\partial}{\partial x_j}(\phi u)(x,t)| \leq C_1 + C'' exp(|x|^\gamma).
$$

So we can choose some constants $C > 0$ and $\gamma < 2$ such that

$$
|\frac{\partial}{\partial x_j} u(x,t)| \leq C exp(|x|^\gamma)
$$

i.e.

$$
\frac{\partial}{\partial x_j} u(x,t) \in \mathcal{E}[0,T].
$$

Proposition 2.2.3 *Let $u(x,t)$ be a non-negative continuous solution of (2.7) in $Q_T = [0,T] \times R^m$. Suppose that $u(x,t)$ is bounded in Q_T. Then*

i)$\nabla_x u$, $\nabla_x \nabla_x u$ *and* u_t *are continuous and bounded in* Q_T,

ii) *u is the regular solution of (2.1) -(2.2) in* $[0,T]$.

Proof. For the proof of boundedness of $\nabla_x u$, it is enough to prove that $|\frac{\partial u}{\partial x_j}|$ is bounded. From Proposition 2.2.1 we get

$$\begin{aligned} |\frac{\partial u}{\partial x_j}(x,t)| &= |\frac{\partial u_0}{\partial x_j}(x,t) + \int_0^t ds \int_{R^m} \frac{\partial}{\partial x_j} H(t-s, x-y) u^{1+\alpha}(y,s) dy| \\ &\leq \int_{R^m} |-\frac{\partial}{\partial y_j} H(t-s,x-y)||a(y)|dy \\ &+ \int_0^t ds \int_{R^m} |-\frac{\partial}{\partial y_j} H(t-s,x-y)||u^{1+\alpha}(y,s)|dy, \end{aligned}$$

since u is bounded, $u^{1+\alpha}$ is also bounded in Q_T and since $a(y) \in \mathcal{A}$, $a(y)$ is also bounded in $\mathbb{R}^m$. Therefore we obtain

$$\begin{aligned} |\frac{\partial u}{\partial x_j}(x,t)| &\leq C_0 \int_{R^m} |\nabla_y H(t-s,x-y)|dy \\ &+ C_1 \int_0^t ds \int_{R^m} |\nabla_y H(t-s,x-y)|dy, \end{aligned}$$

where

$$\begin{aligned} \int_{R^m} |\nabla_y H(t-s,x-y)|dy &\leq C(t-s)^{-(m+1)/2} \int_{R^m} exp(-\frac{|x-y|^2}{9(t-s)})dy, \\ &\leq C(t-s)^{-(m+1)/2}(9(t-s))^{m/2} \int_{R^m} exp(-|\xi|^2)d\xi, \\ &= C(t-s)^{-1/2} \int_{R^m} exp(-|\xi|^2)d\xi, \\ &= C(t-s)^{-1/2}\pi^{m/2} = C_2, \end{aligned}$$

and thus

$$|\frac{\partial u}{\partial x_j}(x,t)| \leq C_3 + C_4 \int_0^t (t-s)^{-1/2} ds = C$$

which states that $\frac{\partial u}{\partial x_j}(x,t)$ is bounded in Q_T. Since $a(x)$ and $u^{1+\alpha}(x,t)$ are continuous in Q_T, $\frac{\partial u}{\partial x_j}(x,t)$ is not continuous only if $t = s$ but, $t \neq s$ otherwise $H(t-s, x-y)$ becomes undefined $(0 \leq s < t \leq T)$. So we can conclude that

$u^\alpha \frac{\partial u}{\partial x_j}(x,t)$ is also continuous and bounded in Q_T, therefore

$$|u^\alpha \frac{\partial u}{\partial x_j}(x,t)| \le C,$$

for some constant $C > 0$.

For the proof of boundedness of $\Delta_x u(x,t)$, it is enough to prove that $\frac{\partial^2 u}{\partial x_i^2}(x,t)$ is bounded. Using integration by parts we get

$$\begin{aligned}
\frac{\partial u}{\partial x_j}(x,t) &= \frac{\partial u_0}{\partial x_j}(x,t) + \int_0^t ds \int_{R^m} \frac{\partial}{\partial x_j} H(t-s, x-y) u^{1+\alpha}(y,s) dy \\
&= \frac{\partial u_0}{\partial x_j}(x,t) - \int_0^t ds \int_{R^m} \frac{\partial}{\partial y_j} H(t-s, x-y) u^{1+\alpha}(y,s) dy \\
&= \frac{\partial u_0}{\partial x_j}(x,t) - \int_0^t \Big\{ H(t-s, x-y) u^{1+\alpha}(s,y) \mid_{\partial R^m} \\
&- (1+\alpha) \int_{R^m} H(t-s, x-y) u^\alpha(y,s) \frac{\partial u}{\partial y_j}(y,s) dy \Big\} ds,
\end{aligned}$$

since $H(t-s, x-y) = 0$ on $\partial \mathbb{R}^m$,

$$\frac{\partial u}{\partial x_j}(x,t) = \frac{\partial u_0}{\partial x_j}(x,t) + (1+\alpha) \int_0^t ds \int_{R^m} H(t-s, x-y) u^\alpha(y,s) \frac{\partial u}{\partial y_j}(y,s) dy. \tag{2.9}$$

On account of the boundedness of $u^\alpha \frac{\partial u}{\partial x_j}(x,t)$, differentiating (2.9) with respect to x_k we get

$$\begin{aligned}
&|\frac{\partial^2 u}{\partial x_k \partial x_j}(x,t)| \le |\frac{\partial^2 u_0}{\partial x_k \partial x_j}(x,t)| \\
&+ (1+\alpha) \int_0^t ds \int_{R^m} |\frac{\partial}{\partial x_k} H(t-s, x-y)||u^\alpha(y,s) \frac{\partial u}{\partial y_j}(y,s)| dy \quad (2.10) \\
&\le |\frac{\partial^2 u_0}{\partial x_k \partial x_j}(x,t)| + C \int_0^t ds \int_{R^m} |\frac{\partial}{\partial y_k} H(t-s, x-y)| dy \\
&\le \int_{R^m} |\frac{\partial^2}{\partial x_k \partial x_j} H(t-s, x-y)||a(y)| dy \\
&+ C \int_0^t ds \int_{R^m} |\nabla_x H(t-s, x-y)| dy
\end{aligned}$$

Now the boundedness of $\int_{R^m} |\nabla_x H(t-s, x-y)|dy$ and $a(y)$ imply

$$
\begin{aligned}
|\frac{\partial^2}{\partial x_i^2} u(x,t)| &\leq C \int_{R^m} |\frac{\partial^2}{\partial x_i^2} H(t-s, x-y)|dy + C', \\
&\leq C \int_{R^m} |\Delta_x H(t-s, x-y)|dy + C', \\
&= C \int_{R^m} |\frac{\partial}{\partial t} H(t-s, x-y)|dy + C', \\
&\leq C \frac{\partial}{\partial t} \int_{R^m} H(t-s, x-y)dy + C', \\
&= C',
\end{aligned}
$$

for $j = k = 1$. So $\frac{\partial^2}{\partial x_i^2} u(x,t)$ is bounded in Q_T. Since $\frac{\partial}{\partial y_j} H(t-s, x-y)$, $\frac{\partial^2}{\partial y_k \partial y_j} H(t-s, x-y)$, $a(y)$ and $u^\alpha \frac{\partial u}{\partial y_j}(y,s)$ are continuous in Q_T, (2.10) gives that $\frac{\partial^2}{\partial x_i^2} u(x,t)$ is also continuous in Q_T.

Next, we claim that $u = u(t,x)$ is Hölder continuous in t in the sense that there is a constant C independent of t and x such that

$$
|u(t+h, x) - u(t,h)| \leq C\sqrt{h}, \qquad (0 \leq t < t+h \leq T, x \in R^m).
$$

First we will show that

$$
u_0(t,x) = \int_{R^m} H(t, x-y)a(y)dy = (4\pi t)^{-m/2} \int_{R^m} exp(-\frac{|x-y|^2}{4t})a(y)dy
$$

is Hölder continuous.

If we let $y = x + 2\sqrt{t}\xi$, then we get

$$
\begin{aligned}
u_0(t,x) &= (4\pi t)^{-m/2}(4t)^{m/2} \int_{R^m} exp(-|\xi|^2)a(x + 2\sqrt{t}\xi)d\xi \\
&= \pi^{-m/2} \int_{R^m} exp(-|\xi|^2)a(x + 2\sqrt{t}\xi)d\xi.
\end{aligned}
$$

In a similar way we find

$$
u_0(t+h, x) = \pi^{-m/2} \int_{R^m} exp(-|\xi|^2)a(x + 2\sqrt{t+h}\xi)d\xi,
$$

which leads to

$$\begin{aligned}|u_0(t+h,x)-u_0(t,x)| &\leq \pi^{-m/2}\int_{R^m} exp(-|\xi|^2)|a(x+2\sqrt{t+h}\xi), \\ &- a(x+2\sqrt{t}\xi)|d\xi.\end{aligned}$$

Since $a(y)\in\mathcal{A}$, $a(y)$ and $\nabla_y a(y)$ are bounded in $\mathbb{R}^m$. So if we apply mean value theorem to $a(y)$, we get

$$\begin{aligned}|a(x+2\sqrt{t+h}\xi)-a(x+2\sqrt{t}\xi)| &= |\nabla_y a(\eta)||x+2\sqrt{t+h}\xi-x-2\sqrt{t}\xi| \\ &\leq 2C|\xi||\sqrt{t+h}-\sqrt{t}| \\ &\leq K|\sqrt{t}+\sqrt{h}-\sqrt{t}| \\ &= K\sqrt{h}.\end{aligned}$$

So

$$\begin{aligned}|u_0(t+h,x)-u_0(t,x)| &\leq K\pi^{-m/2}\sqrt{h}\int_{R^m} exp(-|\xi|^2)d\xi \\ &= K\sqrt{h}\pi^{-m/2}\pi^{m/2}=K\sqrt{h}.\end{aligned}$$

Therefore $u_0(t,x)$ is Hölder continuous in t.

In order to show that the Hölder continuity of $u(t,x)$, which is given by (2.7), it is enough to show that $(\phi u)(t,x)=w(t,x)$ is Hölder continuous. Thus we will evaluate

$$\begin{aligned}I_h &= w(t+h,x)-w(t,x) \\ &= \int_0^{t+h} ds\int_{R^m} H(t+h-s,x-y)u^{1+\alpha}(s,y)dy \\ &- \int_0^{t} ds\int_{R^m} H(t-s,x-y)u^{1+\alpha}(s,y)dy \\ &= \int_t^{t+h} ds\int_{R^m} H(t+h-s,x-y)u^{1+\alpha}(s,y)dy\end{aligned}$$

$$
\begin{aligned}
+ & \int_0^t ds \int_{R^m} H(t+h-s, x-z) u^{1+\alpha}(s,z) dz \\
- & \int_0^t ds \int_{R^m} H(t-s, x-y) u^{1+\alpha}(s,y) dy
\end{aligned}
$$

where we have used

$$
\begin{aligned}
& \int_{R^m} H(t, x-y) H(\gamma, y) dy \\
= \; & (4\pi t)^{-m/2} (4\pi\gamma)^{-m/2} \int_{R^m} exp(-\frac{|x-y|^2}{4t}).exp(-\frac{|y|^2}{4\gamma}) dy.
\end{aligned}
$$

If we let $y = x + 2\sqrt{t}\xi$, we get

$$
\begin{aligned}
& \int_{R^m} H(t, x-y) H(\gamma, y) dy \\
= \; & (4\pi^2\gamma)^{-m/2} \int_{R^m} exp(-|\xi|^2) exp(-\frac{1}{4\gamma}|x + (4t)^{1/2}\xi|^2) d\xi_i \\
= \; & (4\pi^2\gamma)^{-m/2} \prod_{i=1}^m \int_R exp(-|\xi_i|^2) exp(-\frac{1}{4\gamma}|x_i + (4t)^{1/2}\xi_i|^2) d\xi_i \\
= \; & (4\pi^2\gamma)^{-m/2} \prod_{i=1}^m exp(-\frac{|x_i|^2}{4\gamma}) \int_R exp(-\frac{(t+\gamma)}{\gamma}\{\xi_i^2 + \frac{t^{1/2}}{t+\gamma} x_i \xi_i\}) \\
= \; & (4\pi^2\gamma)^{-m/2} e^{-\frac{|x|^2}{4\gamma}} \prod_{i=1}^m \int_R exp(-\frac{(t+\gamma)}{\gamma}\{\xi_i + \frac{t^{1/2}}{2(t+\gamma)} x_i\}^2 \\
+ \; & \frac{t}{4\gamma(t+\gamma)} x_i^2) d\xi_i \\
= \; & (4\pi^2\gamma)^{-m/2} e^{-\frac{|x|^2}{4\gamma}} e^{\frac{t}{4\gamma(t+\gamma)}|x|^2} \prod_{i=1}^m \int_R exp(-\frac{(t+\gamma)}{\gamma}\{\xi_i + \frac{t^{1/2}}{2(t+\gamma)} x_i\}^2) d\xi_i.
\end{aligned}
$$

Using substitution $z_i = \xi_i + \frac{t^{1/2}}{2(t+\gamma)} x_i$, we have

$$
\begin{aligned}
& \int_{R^m} H(t, x-y) H(\gamma, y) dy \\
= \; & (4\pi^2\gamma)^{-m/2} e^{-\frac{-|x|^2}{4(t+\gamma)}} \prod_{i=1}^m \int_R exp(-\frac{(t+\gamma)}{\gamma} z_i^2) dz_i \\
= \; & (4\pi^2\gamma)^{-m/2} e^{-\frac{-|x|^2}{4(t+\gamma)}} \prod_{i=1}^m (\frac{\pi\gamma}{t+\gamma})^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= (4\pi^2\gamma)^{-m/2} e^{-\frac{-|x|^2}{4(t+\gamma)}} (\frac{\pi\gamma}{t+\gamma})^{m/2} \\
&= \{4\pi(t+\gamma)\}^{-m/2} \exp(-\frac{-|x|^2}{4(t+\gamma)}) \\
&= H(t+\gamma, x)
\end{aligned}
$$

and this also yields

$$
\int_{R^m} H(t-s, x-y) H(h, x-y) dy = H(t+h-s, x-z). \tag{2.11}
$$

Now introducing

$$
I_1 = \int_t^{t+h} ds \int_{R^m} H(t+h-s, x-y) u^{1+\alpha}(s,y) dy
$$

and

$$
\nu(s,y) = \int_{R^m} H(h, y-z) u^{1+\alpha}(s,z) dz - u^{1+\alpha}(s,y)
$$

we find

$$
I_h = I_1 + \int_0^t ds \int_{R^m} H(t-s, x-y) \nu(s,y) dy.
$$

For sake of simplicity, letus use the notation

$$
I_2 = \int_0^t ds \int_{R^m} H(t-s, x-y) \nu(s,y) dy.
$$

Hence

$$
I_h = I_1 + I_2.
$$

Because of the boundedness of $u(t,x)$, it is obvious that

$$
\begin{aligned}
|I_1| &= |\int_t^{t+h} ds \int_{R^m} H(t+h-s, x-y) u^{1+\alpha}(s,y) dy| \\
&\leq C \int_t^{t+h} ds \int_{R^m} H(t+h-s, x-y) dy \\
&= C \int_t^{t+h} ds.1 = Ch.
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned} |\nu(s,y)| &= |\int_{R^m} H(h, y-z)u^{1+\alpha}(s,z)dz - u^{1+\alpha}(s,y)| \\ &= |\int_{R^m} (4\pi h)^{-m/2} exp(-\frac{|y-z|^2}{4h})\{u^{1+\alpha}(s,z) - u^{1+\alpha}(s,y)\}dz| \\ &\leq C\sqrt{h} \end{aligned}$$

since $|\nabla_x(u^{1+\alpha}(s,x))| = (1+\alpha)u^\alpha(s,x)|\nabla_x u(s,x)| < C$ in Q_T.

Therefore we get

$$\begin{aligned} |I_2| &\leq \int_0^t ds \int_{R^m} H(t-s, x-y)|\nu(s,y)|dy \\ &\leq C''\sqrt{h}\int_0^t ds \int_{R^m} H(t-s, x-y)dy \\ &= C''\sqrt{h}\int_0^t ds \\ &\leq C''T\sqrt{h} = C_0\sqrt{h}. \end{aligned}$$

Then

$$|w(t+h,x) - w(t,x)| \leq |I_1| + |I_2| \leq Ch + C_0\sqrt{h} \leq C_1\sqrt{h}$$

for some $C_1 > 0$. This will establish the uniform Hölder continuity of $u(t,x)$ in t.

We turn to u_t. Taking a small positive number ϵ, we put

$$(\phi_\epsilon u)(t,x) = \int_0^{t-\epsilon} ds \int_{R^m} H(t-s, x-y)u^{1+\alpha}(s,y)dy, \qquad (\epsilon \leq t \leq T, x \in \mathbb{R}^m).$$

Since $u^{1+\alpha}(t,x)$ is bounded,

$$\begin{aligned} |\phi_\epsilon u(t,x) - \phi u(t,x)| &= |\int_{t-\epsilon}^t ds \int_{R^m} H(t-s, x-y)u^{1+\alpha}(s,y)dy| \\ &\leq \int_{t-\epsilon}^t ds \int_{R^m} H(t-s, x-y)|u^{1+\alpha}(s,y)|dy \end{aligned}$$

$$\begin{aligned} &\leq C\int_{t-\epsilon}^{t} ds \int_{R^m} H(t-s, x-y)dy \\ &= C\int_{t-\epsilon}^{t} ds.1 \\ &= C\epsilon \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Therefore $\phi_\epsilon u(t,x)$ tends to $\phi u(t,x)$ as $\epsilon \to 0$ unifomly in $[\delta, T] \times R^m$, δ being a positive number. Recalling

$$\frac{\partial}{\partial t} H(t-s, x-y) = \Delta_x H(t-s, x-y) = \Delta_y H(t-s, x-y),$$

we have

$$\begin{aligned} \frac{\partial}{\partial t}(\phi_\epsilon u)(t,x) &= \frac{\partial}{\partial t}\int_0^{t-\epsilon} ds \int_{R^m} H(t-s, x-y)u^{1+\alpha}(s,y)dy \\ &= \int_{R^m} H(t-(t-\epsilon), x-y)u^{1+\alpha}(t-\epsilon, y)dy \\ &+ \int_0^{t-\epsilon} ds \int_{R^m} \frac{\partial}{\partial t} H(t-s, x-y)u^{1+\alpha}(s,y)dy \\ &= \int_{R^m} H(\epsilon, x-y)u^{1+\alpha}(t-\epsilon, y)dy \\ &+ \int_0^{t-\epsilon} ds \int_{R^m} \Delta_y H(t-s, x-y)u^{1+\alpha}(s,y)dy, \end{aligned}$$

if we apply the Green's identity to second integral of the above equation, we get

$$\begin{aligned} \int_{R^m} \Delta_y H(t-s, x-y)u^{1+\alpha}(s,y)dy &= \int_{\partial R^m} \Big\{ u^{1+\alpha}(s,y)\frac{\partial}{\partial n} H(t-s, x-y) \\ &- H(t-s, x-y)\frac{\partial}{\partial n} u^{1+\alpha}(s,y) \Big\} dS \\ &+ \int_{R^m} H(t-s, x-y)\Delta_y u^{1+\alpha}(s,y)dy \\ &= \int_{R^m} H(t-s, x-y)\Delta_y u^{1+\alpha}(s,y)dy, \end{aligned}$$

since

$$\int_{\partial R^m} \Big\{ u^{1+\alpha}(s,y)\frac{\partial}{\partial n} H(t-s, x-y) - H(t-s, x-y)\frac{\partial}{\partial n} u^{1+\alpha}(s,y) \Big\} dS = 0.$$

So we get

$$\frac{\partial}{\partial t}(\phi_\epsilon u)(t,x) = \tilde{I}_1 + \tilde{I}_2$$

where

$$\tilde{I}_1 = \int_{R^m} H(\epsilon, x-y) u^{1+\alpha}(t-\epsilon, y)dy$$

and

$$\tilde{I}_2 = \int_0^{t-\epsilon} \int_{R^m} H(t-s, x-y)\Delta_y u^{1+\alpha}(s,y)dy.$$

Then

$$\begin{aligned} |\tilde{I}_1 - u^{1+\alpha}(t,x)| &= |\int_{R^m} H(\epsilon, x-y)u^{1+\alpha}(t-\epsilon,y)dy - u^{1+\alpha}(t,x)| \\ &= |(4\pi\epsilon)^{-m/2}\int_{R^m} exp(-\frac{|x-y|^2}{4\epsilon})u^{1+\alpha}(t-\epsilon,y)dy \\ &- u^{1+\alpha}(t,x)| \end{aligned}$$

if we let $y = x + 2\sqrt{\epsilon}\xi$ and put $\pi^{m/2} = \int_{R^m} exp(|\xi|^2)d\xi$, we get

$$\begin{aligned} |\tilde{I}_1 - u^{1+\alpha}(t,x)| &= \pi^{-m/2}|\int_{R^m} exp(-|\xi|^2)u^{1+\alpha}(t-\epsilon, x+2\sqrt{\epsilon})d\xi \\ &- \int_{R^m} exp(-|\xi|^2)u^{1+\alpha}(t,x)|d\xi| \end{aligned}$$

$$\leq \pi^{-m/2}\int_{R^m} exp(-|\xi|^2)|u^{1+\alpha}(t-\epsilon, x+2\sqrt{\epsilon}) - u^{1+\alpha}(t,x)|d\xi.$$

Since $u^{1+\alpha}(s,y)$ is bounded and uniformly continuous in Q_T,

$$|u^{1+\alpha}(t-\epsilon, x+2\sqrt{\epsilon}) - u^{1+\alpha}(t,x)| \to 0 \ \text{ as } \ \epsilon \to 0.$$

Therefore

$$\tilde{I}_1 \to u^{1+\alpha}(t,x) \ \text{ as } \ \epsilon \to 0.$$

On the other hand, $\tilde{I}_2$ converges uniformly in Q_T to

$$\varphi(t,x) = \int_0^t ds \int_{R^m} H(t-s, x-y)\Delta_y u^{1+\alpha}(s,y)dy,$$

by boundedness of $\Delta_y u^{1+\alpha}(s,y)$ in Q_T. Then

$$\begin{aligned}
|\tilde{I}_2 - \varphi(t,x)| &= |\int_{t-\epsilon}^{t} ds \int_{R^m} H(t-s,x-y)\Delta_y u^{1+\alpha}(s,y)dy| \\
&\leq \int_{t-\epsilon}^{t} ds \int_{R^m} H(t-s,x-y)|\Delta_y u^{1+\alpha}(s,y)|dy \\
&\leq C\int_{t-\epsilon}^{t} ds \int_{R^m} H(t-s,x-y)dy \\
&= C\int_{t-\epsilon}^{t} ds.1 \\
&= C\epsilon \quad \to 0 \ \text{ as } \ \epsilon \to 0.
\end{aligned}$$

At this stage we have

$$\frac{\partial}{\partial t}\phi u(t,x) = u^{1+\alpha}(t,x) + \varphi(t,x)$$

and since

$$\begin{aligned}
\varphi(t,x) &= \int_0^t ds \int_{R^m} H(t-s,x-y)\Delta_y u^{1+\alpha}(s,y)dy \\
&= \int_0^t ds \int_{R^m} \Delta_y H(t-s,x-y)u^{1+\alpha}(s,y)dy \\
&= \int_0^t ds \int_{R^m} \Delta_x H(t-s,x-y)u^{1+\alpha}(s,y)dy \\
&= \Delta_x\Big\{\int_0^t ds \int_{R^m} \Delta_x H(t-s,x-y)u^{1+\alpha}(s,y)dy\Big\} \\
&= \Delta_x \phi u(t,x),
\end{aligned}$$

we get

$$\frac{\partial}{\partial t}\phi u(t,x) = u^{1+\alpha}(t,x) + \Delta_x \phi u(t,x).$$

From the proposition 2.2.1, if we put $\phi u = u - u_0$, then

$$\begin{aligned}
\frac{\partial}{\partial t}(u(t,x) - u_0(t,x)) &= u^{1+\alpha}(t,x) + \Delta(u(t,x) - u_0(t,x)) \\
u_t(t,x) - \frac{\partial}{\partial t}u_0(t,x) &= u^{1+\alpha}(t,x) + \Delta u(t,x) - \Delta u_0(t,x))
\end{aligned}$$

and since

$$\begin{aligned}\frac{\partial}{\partial t}u_0(t,x) &= \int_{R^m}\frac{\partial}{\partial t}H(t,x-y)a(y)dy \\ &= \int_{R^m}\Delta_x H(t,x-y)a(y)dy \\ &= \Delta_x\int_{R^m}H(t,x-y)a(y)dy \\ &= \Delta_x u_0(t,x),\end{aligned}$$

we get

$$u_t(t,x) = u^{1+\alpha}(t,x) + \Delta u(t,x).$$

Thus we established (**i**) and (**ii**) in the proposition.

Now we put

$$\| \nu \| = \sup_{x\in R^m, 0\le t}\frac{\nu(t,x)}{\rho(t,x)}$$

for any function ν with $|\nu| \in \mathcal{L}[\prime,\infty)$ being $H(t+\gamma)$. Then obviously we have

$$|\nu(t,x)| = \| \nu \| \rho(t,x). \tag{2.12}$$

We will study the non-linear integral transformation ϕ defined by

$$(\phi u)(t,x) = \int_0^t ds\int_{R^m}H(t-s,x-y)u^{1+\alpha}(s,y)dy. \tag{2.13}$$

Lemma 2.2.4 *Let $u = u(t,x)$ be a regular solution of (2.1) -(2.2) in $\mathcal{E}$ $[0,T]$ with a nontrivial initial value $a \in \mathcal{A}$. Then we have*

$$J_0^{-\alpha} - u(t,0)^{-\alpha} \ge \alpha t, \qquad (0\le t\le T)$$

where

$$J_0 = J_0(t) = \int_{R^m}H(t,x)a(x)dx.$$

and

$$H(t,x) = (4\pi t)^{-m/2} exp(-\frac{|x|^2}{4t}) \qquad (t > 0, x \in \mathbb{R}^m)$$

is the Green Function of the heat equation.

Proof. Let ε be a positive constant. Fix t in $0 \le t \le T$. Then we put

$$\nu_\varepsilon = \nu_\varepsilon(s,x) = H(t-s+\varepsilon, x), \qquad (0 \le s \le t \le T, x \in \mathbb{R}^m)$$

and

$$J_\varepsilon = J_\varepsilon(s) = \int_{R^m} \nu_\varepsilon(s,x) u(s,x) dx.$$

We can conclude that, ν_ε is regular in $Q_T = [0,T] \times \mathbb{R}^m$ since $H(t,x)$ is regular in Q_T. Also

$$\frac{\partial \nu_\varepsilon}{\partial s} = -\Delta \nu_\varepsilon$$

since $H_t(t,x) = \Delta H(t,x)$.

Since $\nu_\varepsilon(s,x)$ is positive everywhere in $[0,t] \times \mathbb{R}^m$ and $u(s,x)$ is also positive in $(0,t] \times \mathbb{R}^m$, $J_\varepsilon > 0 \ \ \forall s \in [0,t]$, and also u and ν_ε are continuous, thus $J_\varepsilon > 0$ is also continuous in s if it exists.

Since $u \in \mathcal{E}[0,T]$, there are some positive constants M and $\beta < 2$ such that

$$0 \le u(s,x) \le M exp(|x|^\beta), \qquad (0 \le s \le T, x \in \mathbb{R}^m).$$

Now we will prove the existence of J_ε. Let $x = 2(t-s+\varepsilon)^{1/2}\eta$ and $dx = 2^m(t-s+\varepsilon)^{m/2}d\eta)$, then we have

$$\begin{aligned} 0 \le J_\varepsilon(s) &= \int_{R^m} H(t-s+\varepsilon) u(s,x) dx \\ &\le M \int_{R^m} H(t-s+\varepsilon, x) exp(|x|^\beta) dx \end{aligned}$$

$$
\begin{aligned}
&= M\pi^{-m/2}\int_{R^m} exp(-|\eta|^2)exp(2^{\beta}(t-s+\varepsilon)^{\frac{\beta}{2}}|\eta|^{\beta})d\eta \\
&\leq M\pi^{-m/2}\int_{R^m} exp(-|\eta|^2+2^{\beta}(t+\varepsilon)^{\frac{\beta}{2}}|\eta|^{\beta}) \\
&\leq M\pi^{-m/2}\int_{R^m} exp(-|\eta|^2+\gamma|\eta|^{\beta})d\eta.
\end{aligned}
$$

Now claim that $I_m = \int_{R^m} exp(-|\eta|^2+\gamma|\eta|^{\beta})d\eta < \infty$. Since

$$
|\eta|^{\beta} = \{\sum_{i=1}^{m}|\eta_i|^2\}^{\beta/2} \leq \sum_{i=1}^{m}|\eta_i|^{\beta} \qquad \text{for} \qquad \beta/2 < 1
$$

we have

$$
-|\eta|^2+\gamma|\eta|^{\beta} \leq -\sum_{i=1}^{m}|\eta_i|^2+\gamma\sum_{i=1}^{m}|\eta_i|^{\beta} = \sum_{i=1}^{m}(-|\eta_i|^2+\gamma|\eta_i|^{\beta})
$$

Then

$$
\begin{aligned}
I_m &\leq \int_{R^m} exp(\sum_{i=1}^{m}(-|\eta_i|^2+\gamma|\eta_i|^{\beta})d\eta \\
&= \prod_{i=1}^{m}\int_{R} exp(-|\eta_i|^2+\gamma|\eta_i|^{\beta})d\eta_i \\
&= 2^m\prod_{i=1}^{m}\int_0^{\infty} exp(-\eta_i^2+\gamma\eta_i^{\beta})d\eta_i.
\end{aligned}
$$

Since $\int_0^1 exp(-\eta_i^2+\gamma\eta_i^{\beta})d\eta_i = c$ for some constant $c > 0$, it is enough to prove the convergence of $\int_1^{\infty} exp(-\eta_i^2+\gamma\eta_i^{\beta})d\eta_i$. Since

$$
\lim_{\eta_i\to\infty}\frac{exp(-\eta_i^2+\gamma\eta_i^{\beta})}{1/\eta_i^2} = 0 \qquad \text{and} \qquad \int_1^{\infty}\frac{d\eta_i}{\eta_i^2} \qquad \text{is convergent,}
$$

so by limit comparison test

$$
\int_1^{\infty} exp(-\eta_i^2+\gamma\eta_i^{\beta})d\eta_i
$$

also converges. Therefore

$$\int_0^\infty exp(-\eta_i^2 + \gamma\eta_i^\beta)d\eta_i \le C < \infty,$$

for some constant $C > 0$. So $I_m \le 2^m C^m < \infty$. Therefore J_ε exists uniformly and is a continuous function of s.

Furthermore, J_ε is continuously differentiable and satisfies the following equation

$$\frac{dJ_\varepsilon(s)}{ds} = \int_{R^m} \nu_\varepsilon(s,x)u(s,x)^{1+\alpha}dx. \tag{2.14}$$

We put

$$J^{(N)}(s) = \int_{R^m} \nu_\varepsilon.u.\rho_N dx \qquad (N = 1, 2, ..).$$

where ρ_N as in introduced in (2.3).

Now claim that $J^{(N)}(s)$ tends to $J_\varepsilon(s)$ as N $\to \infty$ uniformly in s;

$$|J^{(N)}(s) - J_\varepsilon(s)| = |\int_{R^m} \nu_\varepsilon.u.\rho_N dx - \int_{R^m} \nu_\varepsilon.udx| \le \int_{R^m} \nu_\varepsilon.u|\rho_N - 1|dx \to 0$$

as $N \to \infty$. Since as $N \to \infty$, x/N converges to center of the ball in R^m (i.e. $|x/N| < 1$), $\rho_N(x) = \rho(x/N) \to 1$ as $N \to \infty$. Therefore

$$|\rho_N(x) - 1| \to 0 \qquad \text{as} \qquad N \to \infty.$$

So we can differentiate $J^{(N)}(s)$,

$$\begin{aligned}
\frac{d}{ds}J^{(N)}(s) &= \int_{R^m} \frac{d}{ds}(\nu_\varepsilon.u)\rho_N(x)dx \\
&= \int_{R^m} (\frac{d}{ds}\nu_\varepsilon.u + \nu_\varepsilon.\frac{d}{ds}u)\rho_N(x)dx \\
&= \int_{R^m} (-\Delta\nu_\varepsilon.u + \nu_\varepsilon.\Delta u).\rho_N dx + \int_{R^m} \nu_\varepsilon.u^{1+\alpha}.\rho_N dx. \\
&=: I_1 + I_2
\end{aligned}$$

First let us take I_2:

$$\begin{aligned} |I_2 - \int_{R^m} \nu_\varepsilon(s,x)u(s,x)^{1+\alpha}dx| &= |\int_{R^m} \nu_\varepsilon(s,x)u(s,x)^{1+\alpha}.\rho_N dx \\ &- \int_{R^m} \nu_\varepsilon(s,x)u(s,x)^{1+\alpha}dx| \\ &\leq \int_{R^m} \nu_\varepsilon(s,x)u(s,x)^{1+\alpha}|\rho_N(x)-1|dx \end{aligned}$$

which tends to 0 as $N \to \infty$. Thus I_2 tends to right hand side of (2.14). Now let us prove that $I_1 \to 0$ as $N \to \infty$.

$$\begin{aligned} I_1 &= \int_{R^m} (-\Delta\nu_\varepsilon.u + \nu_\varepsilon.\Delta u)\rho_N dx \\ &= -\int_{R^m} \Delta\nu_\varepsilon.(u\rho_N)dx + \int_{R^m} \nu_\varepsilon.\Delta u.\rho_N dx \end{aligned}$$

if we use Green‘s formula for the first part of the above integral, then we get

$$\begin{aligned} I_1 &= -\int_{R^m} \nu_\varepsilon.\Delta(u\rho_N)dx - \int_{\partial R^m} (u.\rho_N \frac{\partial \nu_\varepsilon}{\partial n} - \nu_\varepsilon \frac{\partial(u\rho_N)}{\partial n})d\sigma \\ &+ \int_{R^m} \nu_\varepsilon.\Delta u.\rho_N dx. \end{aligned}$$

Let us $B(0,\delta)$ be a ball in $\mathbb{R}^m$. Then

$$\nu_\varepsilon \mid_{\partial B(0,\delta)} = H(t-s+\varepsilon,\delta) = [4\pi(t-s+\varepsilon)]^{-m/2} exp(-\frac{\delta^2}{4(t-s+\varepsilon)})$$

tends to 0 as $\delta \to \infty$. Therefore $\nu_\varepsilon \mid_{\partial R^m} = 0$ and

$$\rho_N(x) \mid_{\partial B(0,\delta)} = \rho_N(\delta) = \rho(\frac{\delta}{N})$$

tends to 0as $\delta \to \infty$, for a fixed N, $|\delta| > N$. Therefore $\rho_N(x) \mid_{\partial R^m} = 0$. Thus

$$I_1 = -2\int_{R^m} \nu_\varepsilon.\nabla u.\nabla\rho_N dx - \int \nu_\varepsilon.u.\Delta\rho_N dx.$$

Let

$$p_N \equiv |\int_{R^m} \nu_\varepsilon.\nabla u.\nabla \rho_N dx| \leq \int_{R^m} \nu_\varepsilon.|\nabla u|.|\nabla \rho_N| dx.$$

From proposition (2.2.2) we have $\frac{\partial u}{\partial x_i} \in \mathcal{E}[0,T], |\nabla u| \in \mathcal{E}[0,T]$. Therefore we have

$$\int_{R^m} \nu_\varepsilon |\nabla u| dx \leq C < \infty$$

as in the proof of the existence of the J_ε.

Now claim that $|\nabla \rho_N| \leq CN^{-1}$ and $|\Delta \rho_N| \leq CN^{-2}$ for some positive constant C. It is easy to see that $|\nabla \rho_N| \leq CN^{-1}$ and $|\Delta \rho_N| \leq CN^{-2}$ holds. Therefore $\rho_N \leq CN^{-1} \to 0$ as $N \to \infty$. We have also

$$\begin{aligned} q_N \equiv |\int_{R^m} \nu_\varepsilon.u.\Delta\rho_N dx| &\leq \int_{R^m} \nu_\varepsilon.u.|\Delta \rho_N| dx \\ &\leq CN^{-2} \int_{R^m} \nu_\varepsilon.u dx \\ &= CN^{-2} J_\varepsilon(s) \\ &\leq CN^{-2} \to 0 \quad \text{as} \quad N \to \infty. \end{aligned}$$

As a consequence

$$\frac{d}{sd} J^N(s) \to \frac{d}{ds} J_\varepsilon(s) \ \text{ uniformly as } \ N \to \infty.$$

Then by Jensen's inequality [[6], page 151] ($u^{1+\alpha}$ is the convex function of u, since $\alpha > 0$)

$$\begin{aligned} \frac{d}{ds} J_\varepsilon(s) &= \int_{R^m} \nu_\varepsilon(s,x) u(s,x)^{1+\alpha} dx \\ &\geq \left\{ \frac{\int_{R^m} \nu_\varepsilon u dx}{\int_{R^m} \nu_\varepsilon dx} \right\}^{1+\alpha} . \int_{R^m} \nu_\varepsilon dx \\ &= (\int_{R^m} \nu_\varepsilon.u dx)^{1+\alpha} \\ &= J_\varepsilon^{1+\alpha}(s). \end{aligned}$$

On the other hand

$$\begin{aligned}\int_{R^m} \nu_\varepsilon(s,x)dx &= \int_{R^m} H(t-s+\varepsilon,x)dx \\ &= [4\pi(t-s+\varepsilon)]^{-m/2}\int_{R^m} exp(-\frac{|x|^2}{4(t-s+\varepsilon)})dx \\ &= \pi^{-m/2}\pi^{m/2} = 1.\end{aligned}$$

Hence

$$\frac{d}{ds}J_\varepsilon(s) \geq J_\varepsilon(s)^{1+\alpha}, \qquad (0 \leq s \leq t).$$

Then $J_\varepsilon(s)^{-1-\alpha}dJ_\varepsilon(s) \geq ds$ or,

$$\int_0^t J_\varepsilon(s)^{-1-\alpha}dJ_\varepsilon(s) \geq \int_0^t ds$$

$$J_\varepsilon^{-\alpha}(0) - J_\varepsilon^{-\alpha}(t) \geq \alpha t. \tag{2.15}$$

A direct computation gives that

$$\begin{aligned}|J_\varepsilon(0) - J_0(0)| &= |\int_{R^m} H(t+\varepsilon,x)u(0,x)dx - \int_{R^m} H(t,x)u(0,x)dx| \\ &\leq \int_{R^m} |H(t+\varepsilon,x) - H(t,x)|a(x)dx \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

or $J_\varepsilon(0) \to J_0$. Now we claim that $J_\varepsilon(t) \to u(t,0)$ as $\varepsilon \to 0$;

$$\begin{aligned}|J_\varepsilon(t) - u(t,0)| &= |\int_{R^m} H(\varepsilon,x)u(t,x)dx - u(0,x)\int_{R^m} H(\varepsilon,x)dx| \\ &\leq \int_{R^m} H(\varepsilon,x)|u(t,x) - u(t,0)|dx \to 0 \quad \text{as} \quad \varepsilon \to 0\end{aligned}$$

since $H(\varepsilon,x) \to 0$ as $\varepsilon \to 0$. Therefore $J_\varepsilon(t) \to u(t,0)$ as $\varepsilon \to 0$. If we take the limit of both side of the inequality (2.15), then we get

$$J_0^{-\alpha} - u(t,0)^{-\alpha} \geq \alpha t. \tag{2.16}$$

This completes the proof of the lemma.

Theorem 2.2.5 *[4] Let $0 < m\alpha < 2$. Suppose that $a \in \mathcal{A}$ does not vanish identically. Then there exists no global solution of (2.1)-(2.2) in $\mathcal{E}[0,\infty)$.*

Proof. *Suppose that there exists a global solution of (2.1)-(2.2) in $\mathcal{E}[0,\infty)$.*

From Lemma 2.2.4

$$J_0^{-\alpha} \geq u(t,0)^{-\alpha} + \alpha t \geq \alpha t,$$

where $J_0 = \int_{R^M} H(t,x)a(x)dx$. Without loss of generality we can can assume that $a(x) > 0$ in a neighborhood of origin. Then we can choose positive constants γ and δ such that $|x| < 2\delta$ implies $a(x) > \gamma$. Now we restrict $t \geq \delta^2$. Then,

$$\begin{aligned} J_0 &= (4\pi t)^{-m/2}\int_{R^m} exp(-\frac{|x|^2}{4t})a(x)dx \\ &\geq (4\pi t)^{-m/2}\gamma\int_{|x|<2\delta} exp(-\frac{|x|^2}{4t})dx. \end{aligned}$$

Since $|x| < 2\delta$ and $\delta^2/t \leq 1$, we get

$$J_0 \geq \gamma exp(-1)(4\pi t)^{-m/2}\int_{|x|<2\delta} dx = C_2 t^{-m/2}.$$

Consequently,

$$C_2 t^{-m/2} \leq J_0 \leq C_1 t^{-1/\alpha} \qquad \text{or} \qquad t^{m\alpha/2-1} \geq (C_1/C_2)^\alpha = C^\alpha > 0$$

but the above inequality is impossible for a large t, because by our assumption $0 < m\alpha < 2$ (i.e. $m\alpha/2 - 1 < 0$). This completes the proof of Theorem 2.2.5.

2.3 Global Existence of Solutions

Under suitable conditions we may have the existence of global solutions. First of all, suppose $a \in \mathcal{A}$ is subject to

$$0 \leq a(x) \leq \delta H(\gamma, x)$$

for some positive constants γ and δ. We fix γ here and will determine δ later.

$$u(t,x) = u_0(t,x) + \int_0^t ds \int_{R^m} H(t-s, x-y) u^{1+\alpha}(s,y) dy \tag{2.17}$$

where

$$u_0(t,x) = \int_{R^m} H(t, x-y) a(y) dy \tag{2.18}$$

is the integral equation asociated with (2.1)-(2.2). We are going to solve (2.17)-(2.18) by iteration in class $\mathcal{L}[0,\infty)$.[4]

Lemma 2.3.1 *Let $m\alpha > 2$. Then $\phi\rho(t,x) \in \mathcal{L}[0,\infty)$ and*

$$\| \phi\rho(t,x) \| \leq c_0.$$

where $\rho(t,x) = H(t+\gamma, x)$ and c_0 is constant given by

$$c_0 = (4\pi)^{-m\alpha/2} \int_0^\infty (s+\gamma)^{-m\alpha/2} ds = \frac{2\gamma}{2-m\alpha} (4\pi\gamma)^{-m\alpha/2}.$$

Proof. From the equation (2.13), $\phi\rho(t,x) \geq 0$ is obvious and the continuity of $\phi H(t+\gamma, x)$ is clear, since γ, t are positive. We note

$$\begin{aligned} \rho^\alpha(s,y) &= H^\alpha(s+\gamma, y), \\ &= (4\pi(s+\gamma))^{-m\alpha/2} exp(-\frac{\alpha|y|^2}{4(s+\gamma)}), \\ &\leq (4\pi(s+\gamma))^{-m\alpha/2}. \end{aligned}$$

This yields

$$\begin{aligned} 0 \leq \phi\rho(t,x) &= \int_0^t ds \int_{R^m} H(t-s, x-y) \rho^{1+\alpha}(s,y) dy \\ &= \int_0^t ds \int_{R^m} H(t-s, x-y) H(s+\gamma, y) \rho^\alpha(s,y) dy \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t ds \int_{R^m} H(t-s, x-y) H(s+\gamma, y)(4\pi(s+\gamma))^{-m\alpha/2} dy \\
&= \int_0^t (4\pi(s+\gamma))^{-m\alpha/2} ds \int_{R^m} H(t-s, x-y) H(s+\gamma, y) dy \\
&\leq H(t+\gamma, x)(4\pi)^{-m\alpha/2} \int_0^\infty (s+\gamma)^{-m\alpha/2} ds \\
&= c_0 \rho(t, x).
\end{aligned}
$$

So we have

$$
\| \phi\rho(t,x) \| = \sup_{x \in R^m, t \geq 0} \frac{|\phi\rho(t,x)|}{\rho(t,x)} \leq c_0
$$

which completes the proof.

Lemma 2.3.2 *Let $m\alpha > 2$ and let $u \in \mathcal{L}[0,\infty)$. Then $\phi u \in \mathcal{L}[0,\infty)$ and we have*

$$
\| \phi u \| \leq c_0 \| u \|^{1+\alpha} .
$$

Proof.

$$
\begin{aligned}
|\phi u(t,x)| &= |\int_0^t ds \int_{R^m} H(t-s, x-y) u^{1+\alpha}(s,y) dy|, \\
&\leq \int_0^t ds \int_{R^m} H(t-s, x-y) |u(s,y)|^{1+\alpha} dy.
\end{aligned}
$$

From the inequality (2.12) and Lemma 2.3.1, we have

$$
\begin{aligned}
|\phi u(t,x)| &\leq \| u(t,x) \|^{1+\alpha} \int_0^t ds \int_{R^m} H(t-s, x-y) \rho^{1+\alpha}(s,y) dy \\
&= \| u(t,x) \|^{1+\alpha} \phi\rho(t,x) \\
&\leq c_0 \| u(t,x) \|^{1+\alpha} \rho(t,x).
\end{aligned}
$$

So

$$
\frac{|\phi u(t,x)|}{\rho(t,x)} \leq c_0 \| u(t,x) \|^{1+\alpha}
$$

and

$$\| \phi u(t,x) \| = \sup_{x\in R^m, t\geq 0} \frac{|\phi u(t,x)|}{\rho(t,x)} \leq c_0 \| u(t,x) \|^{1+\alpha} .$$

Lemma 2.3.3 *Let $m\alpha > 2$. Suppose that $u, v \in \mathcal{L}[0,\infty)$ and they satisfy*

$$\| u \| \leq M \qquad \text{and} \qquad \| v \| \leq M$$

for a positive number M. Then we have

$$\| \phi u - \phi v \| \leq c_0(1+\alpha)M^\alpha \| u - v \| .$$

Proof. First of all we will show that

$$|p^{1+\alpha} - q^{1+\alpha}| \leq (1+\alpha)|p-q|.max\{p^\alpha, q^\alpha\}, \qquad (p \geq 0, q \geq 0).$$

Let us assume $p < q$ and apply mean value theorem to $f(x) = x^{1+\alpha}$ on $[p,q]$, then

$$\frac{p^{1+\alpha} - q^{1+\alpha}}{p-q} = (1+\alpha)\xi^\alpha$$

for some $\xi \in (p,q)$, and thus for arbitrary $p \geq 0$, $q \geq 0$ we have

$$|p^{1+\alpha} - q^{1+\alpha}| \leq (1+\alpha)|p-q|\xi^\alpha \leq (1+\alpha)|p-q|.max\{p^\alpha, q^\alpha\}.$$

Putting $p = u(t,x)$ and $q = v(t,x)$ we get

$$|u^{1+\alpha}(t,x) - v^{1+\alpha}(t,x)| \leq (1+\alpha)|u(t,x) - v(t,x)|.max\{u^\alpha(t,x), v^\alpha(t,x)\}.$$

Since $\| u \| \leq M$ and $\| v \| \leq M$, $|u(t,x)| \leq M\rho(t,x)$ and $|v(t,x)| \leq M\rho(t,x)$. Therefore

$$max\{u^\alpha(t,x), v^\alpha(t,x)\} \leq M^\alpha \rho^\alpha(t,x).$$

Hence we get

$$\begin{aligned} |u^{1+\alpha}(t,x) - v^{1+\alpha}(t,x)| &\leq (1+\alpha)|u(t,x) - v(t,x)|M^\alpha \rho^\alpha(t,x) \\ &\leq (1+\alpha)M^\alpha \rho^{1+\alpha}(t,x) \parallel u(t,x) - v(t,x) \parallel . \end{aligned}$$

Then

$$\begin{aligned} |\phi u(t,x) - \phi v(t,x)| &= |\int_0^t ds \int_{R^m} H(t-s,x-y) u^{1+\alpha}(s,y)dy \\ &- \int_0^t ds \int_{R^m} H(t-s,x-y) v^{1+\alpha}(s,y)dy| \\ &\leq \int_0^t ds \int_{R^m} H(t-s,x-y)|u^{1+\alpha}(s,y) - v^{1+\alpha}(s,y)|dy \\ &\leq (1+\alpha)M^\alpha \parallel u(t,x) - v(t,x) \parallel \\ &. \int_0^t ds \int_{R^m} H(t-s,x-y)\rho^{1+\alpha}(s,y)dy \\ &= (1+\alpha)M^\alpha \parallel u(t,x) - v(t,x) \parallel \phi\rho(t,x) \\ &\leq c_0(1+\alpha)M^\alpha \parallel u(t,x) - v(t,x) \parallel \rho(t,x). \end{aligned}$$

This inequality gives us

$$\begin{aligned} \parallel \phi u(t,x) - \phi v(t,x) \parallel &= \sup_{x\in R^m, t\geq 0} \frac{|\phi u(t,x) - \phi v(t,x)|}{\rho(t,x)} \\ &\leq c_0(1+\alpha)M^\alpha \parallel u(t,x) - v(t,x) \parallel \end{aligned}$$

which completes the proof of the Lemma 2.3.3.

Lemma 2.3.4 *Let $2 < m\alpha$. Take any positive number γ. Then there exists a positive number δ with the following property: if $a \in \mathcal{A}$ and $0 \leq \delta H(\gamma, x)$, then there exists a solution of (2.7) in $\mathcal{L}[0,\infty)$, which is subject to*

$$0 \leq a(x) \leq MH(t+\gamma, x) \qquad (t \geq 0, x \in R^m$$

for some positive constant M.

Proof. We will use the the iteration defined by

$$u_{n+1} = u_0 + \phi u_n, \qquad (n = 0, 1, 2, ...) \tag{2.19}$$

with $m\alpha > 2$. According to Lemma 2.3.2, we can continue the iteration indefinitely within $\mathcal{L}[0,\infty)$ and get

So $\parallel u_n \parallel$ is bounded for sufficiently small δ. This implies that $\parallel u_n \parallel\leq M$, $(n = 1, 2, ...)$ for some constant $M = M(\delta)$ in which $M(\delta) \to 0$ as $\delta \to 0$

$$\parallel u_n \parallel = \delta + c_0\gamma_{n-1}^{1+\alpha}(\delta).$$

On the other hand from Lemma 2.3.3,

$$\begin{aligned}
\parallel u_{n+1} - u_n \parallel &= \parallel u_0 + \phi u_n - u_0 - \phi u_{n-1} \parallel \\
&= \parallel \phi u_n - \phi u_{n-1} \parallel \\
&\leq c_0(1+\alpha)M^\alpha(\delta) \parallel u_n - u_{n-1} \parallel .
\end{aligned}$$

We can choose δ such that

$$r = c_0(1+\alpha)M^\alpha(\delta) < 1.$$

Then

$$\begin{aligned}
\parallel u_{n+1} - u_n \parallel &\leq r \parallel u_n - u_{n-1} \parallel \\
&\leq r^2 \parallel u_{n-1} - u_{n-2} \parallel \\
&. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \\
&\leq r^n \parallel u_1 - u_0 \parallel \\
&= r^n \parallel \phi u_0 \parallel \\
&\leq c_0 r^n \parallel u_0 \parallel^{1+\alpha} \\
&\leq c_0 r^n \delta^{1+\alpha}
\end{aligned}$$

which implies that

$$\sum_{n=0}^{\infty} \| u_{n+1} - u_n \| \leq c_0 \delta^{1+\alpha} \sum_{n=0}^{\infty} r^n = \frac{c_0 \delta^{1+\alpha}}{1-r} < \infty.$$

So the series

$$\sum_{n=0}^{\infty} \| u_{n+1} - u_n \|$$

is convergent. Thus u_n converges with respect to the norm $\| \, . \, \|$, that is, u_n/ρ converges uniformly in $[0, \infty) \times R^m$. Hence there exists a function $u(t, x) \in \mathcal{L}[0, \infty)$ such that

$$\| u_n - u \| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.20}$$

Using (2.19) and (2.20), we easily get that u is a solution of (2.7).

Theorem 2.3.5 *[4], [11] Let $2 < m\alpha$. Take any positive number γ. Then there exists a positive number δ with the following property: if $a \in \mathcal{A}$ and $0 \leq \delta H(\gamma, x)$, then there exists a global solution of $u = u(t, x)$ in $\mathcal{E}[0, \infty)$, which is subject to*

$$0 \leq a(x) \leq MH(t + \gamma, x), \qquad (t \geq 0, x \in R^m$$

for some positive constant M.

Proof. The solution we have constructed above is the required solution of (2.1)-(2.2), since it is regular by the Proposition 2.2.3 and since $\mathcal{L}[0, \infty) \subset \mathcal{E}[0, \infty)$.

CHAPTER 3

CAUCHY PROBLEM FOR QUASILINEAR EQUATIONS

3.1 Introduction:

For general nonlinear dissipative operators $A(u)$, there is no known result for the equation

$$u_t = A(u) + u^p.$$

In this chapter we shall look at a special case of the operator $A(u)$. We begin with the following initial value problem:

$$\begin{aligned} u_t &= \operatorname{div}\left\{\frac{\nabla_x(u)}{[1+|\nabla_x(u)|^2]^{1/2}}\right\} + u^p, \qquad x \in \mathbb{R}^N, \quad t > 0 \\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad x \in \mathbb{R}^N. \end{aligned}$$

("Mean Curvature.") Again the interest is in nonnegative solutions.

3.2 Global Existence and Nonexistence of Solutions

In [9], a more general problem is considered, namely ("Generalized Mean Curvature")

$$u_t = \Phi(u) + u^p, \qquad (x,t) \in \mathbb{R}^N \times (0,T) \tag{3.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N \tag{3.2}$$

where

$$\Phi(u) = \operatorname{div}\{\psi[(1+|\nabla u|^2)^{1/2}]\nabla u\},$$

with

$$(i) \quad \psi \in C^{1,\alpha}([1,\infty)); \quad \psi(1) = \psi_0 > 0,$$

$$(ii) \quad 0 \le \psi'(s) + \psi(s) \le (1+\theta)\psi(s) \quad (\theta > 0),$$

$$(iii) \quad \psi(s) \le \psi_M, \quad \psi_M < \infty.$$

Then the critical exponent for (3.1)-(3.2) is $p_c = 1 + 2/N$. More precisely we have the following theorem.

Theorem 3.2.1 *Consider the problem (3.1)-(3.2). Under the above conditions on ψ, with $p_c = 1 + 2/N$, we have:* **(A)** *no positive global solutions if $1 < p < p_c$ and* **(B)** *some global, positive solutions if $p > p_c$. Indeed, for $p > p_c$, for all $w_0, t_0 > 0$, there is $\alpha_0 > 0$ such that if $u_0 \le \alpha_0 w_0 exp(-|x|^2/4t_0 w_0)$, then $u(x,t) \le w_0(t+t_0)^{-N/2} exp(-|x|^2/4(t+t_0)\psi_0)$.*

Proof.(A) We assume that $1 < p < 1 + 2/N$. We may assume u_0 is radially symmetric and decreasing. Let $r = |x|$, then $|\nabla u|^2 = u_r^2$ and

$$\begin{aligned}
\Phi(u) &= \operatorname{div}\{\psi[(1+|\nabla u|^2)^{1/2}]\} \\
&= N r^{-1} u_r \psi(\sqrt{1+u_r^2}) + r\{r^{-1} u_r \psi(\sqrt{1+u_r^2})\}', \\
&= r^{-(N-1)}\Big\{(N-1) r^{N-2} u_r \psi(\sqrt{1+u_r^2}) + r^{N-1}(u_r \psi(\sqrt{1+u_r^2}))'\Big\}, \\
&= r^{-(N-1)}\{r^{N-1} u_r \psi(\sqrt{1+u_r^2})\}',
\end{aligned}$$

so the differential equation (3.1)-(3.2) takes the form

$$u_t = r^{-(N-1)}\{r^{N-1} u_r \psi(\sqrt{1+u_r^2})\}_r + u^p.$$

Let

$$F(t) = \int_0^\infty u(r,t)\varphi(r) r^{N-1} dr$$

where

$$\varphi(r) = C^{-1}e^{-kr^2} \qquad \text{and} \qquad C = \int_0^\infty r^{N-1}e^{-kr^2}dr = \frac{1}{2}\Gamma(N/2)k^{-N/2}.$$

We prove the theorem by showing that F blows up in finite time for some $k > 0$. So

$$\begin{aligned} F'(t) &= \int_0^\infty u_t(r,t)\varphi(r)r^{N-1}dr, \\ &= \int_0^\infty \{r^{N-1}u_r\psi(\sqrt{1+u_r^2})\}_r\varphi(r)dr + \int_0^\infty u^p(r,t)\varphi(r)r^{N-1}dr, \end{aligned}$$

using integration by parts for the first part of the integral above, we get

$$\begin{aligned} F'(t) &= -\int_0^\infty r^{N-1}u_r\psi(\sqrt{1+u_r^2})\varphi_r(r)dr + \int_0^\infty r^{N-1}u^p(r,t)\varphi(r)dr, \\ &:= I_1 + I_2. \end{aligned}$$

Since $\psi(s) \leq \psi_M$, $u_r \leq 0$ and

$$\varphi_r(r) = C^{-1}e^{-kr^2}(-2kr) = -2kr\varphi(r) \leq 0,$$

we have

$$\begin{aligned} I_1 &= -\int_0^\infty r^{N-1}u_r\psi(\sqrt{1+u_r^2})\varphi_r(r)dr, \\ &\geq 2k\psi_M\int_0^\infty r^N u_r\varphi(r)dr, \\ &= -2k\psi_M\int_0^\infty u(r,t)\{Nr^{N-1}\varphi(r) + r^N\varphi_r(r)\}dr, \\ &\geq -2kN\psi_M\int_0^\infty u(r,t)r^{N-1}\varphi(r)dr, \\ &= -2kN\psi_M F(t), \end{aligned}$$

Applying Jensen's Inequality to I_2 we get

$$\begin{aligned} I_2 &= \int_0^\infty r^{N-1}u^p(r,t)\varphi(r)dr, \\ &\geq \left\{\int_0^\infty r^{N-1}u(r,t)\varphi(r)dr\right\}^p, \\ &= F^p(t). \end{aligned}$$

At the end we have

$$F'(t) \geq -2kN\psi_M F(t) + F^p(t).$$

$F(t)$ blows up in finite time if k is such that $F^{p-1}(0) > 2kN\psi_M$. Using this inequality we get

$$\begin{aligned} F'(t) \;\; &\geq \;\; -2kN\psi_M F(t) + F^p(t) > -F^{p-1}(0)F(t) + F^p(t), \\ & 1 < \frac{F'(t)}{F^p(t) - F^{p-1}(0)F(t)}, \\ & \int_0^T dt < \int_0^T \frac{dF(t)}{F^p(t) - F^{p-1}(0)F(t)}, \\ & T < \int_{F(0)}^{F(T)} \frac{d\sigma}{\sigma^p - F^{p-1}(0)\sigma}. \end{aligned}$$

If $F(t)$ blows up at T, then

$$T < \int_{F(0)}^{F(T)} \frac{d\sigma}{\sigma^p - F(0)^{p-1}\sigma} = \int_{F(0)}^{\infty} \frac{d\sigma}{\sigma^p - F(0)^{p-1}\sigma} < \infty.$$

So $F(t)$ has a finite blowing up time if

$$2kN\psi_M < F^{p-1}(0)$$

which leads to

$$2kN\psi_M < \left\{ (2k^{N/2}/\Gamma(N/2)) \int_0^\infty u_0(r) e^{-kr^2} r^{N-1} dr \right\}^{p-1}.$$

Thus in the case of $1 < p < 1 + 2/N$,

$$2N\psi_M k^{1-\frac{N}{2}(p-1)} < \left\{ (2/\Gamma(N/2)) \int_0^\infty u_0(r) e^{-kr^2} r^{N-1} dr \right\}^{p-1}$$

will hold if k is sufficiently small.

B. We assume $p > 1 + 2/N$. Let $\omega_0, t_0 > 0$. Define

$$w(x,t) = \omega_0 (t+t_0)^{-N/2} exp(-|x|^2/4(t+t_0)\psi_0).$$

Then $w(x,t)$ satisfies the differential equation $w_t = \psi_0 \Delta w$. It is sufficient to show that there is a supersolution $\bar{u} \leq w$. Let $\bar{u} = \bar{u}(r,t) = \alpha(t) w(r,t)$ and

$$A(s) = \psi(\sqrt{1+s^2}).$$

Then

$$\begin{aligned}
\Phi(\bar{u}) &= r^{-(N-1)} \Big\{ r^{N-1} A(\bar{u}_r)\bar{u}_r \Big\}_r, \\
&= A(\bar{u}_r)\Delta_r \bar{u} + A'(\bar{u}_r)\bar{u}_r \bar{u}_{rr}, \\
&= \{A(\bar{u}_r) + A'(\bar{u}_r)\bar{u}_r\}\Delta_r \bar{u} - A'(\bar{u}_r)\bar{u}_r\{\bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r\} + A'(\bar{u}_r)\bar{u}_r\bar{u}_{rr}, \\
&= \{A(\bar{u}_r) + A'(\bar{u}_r)\bar{u}_r\}\Delta_r \bar{u} - \frac{N-1}{r} A'(\bar{u}_r)\bar{u}_r^2.
\end{aligned}$$

We know that $\bar{u}$ is a supersolution if

$$\bar{u}_t \geq \Phi(\bar{u}) + \bar{u}^p$$

therefore

$$\alpha'(t)w + \alpha(t)w_t \geq \{A(\bar{u}_r) + A'(\bar{u}_r)\bar{u}_r\}\Delta_r \bar{u} - \frac{N-1}{r} A'(\bar{u}_r)\bar{u}_r^2 + \bar{u}^p$$

and

$$\alpha'(t)w + \alpha(t)w_t \geq \{A(\bar{u}_r) + A'(\bar{u}_r)\bar{u}_r\}\alpha(t)\Delta_r w - \frac{N-1}{r} A'(\bar{u}_r)\alpha^2(t)w_r^2 + \bar{u}^p.$$

This inequality gives

$$\alpha'(t)w \geq \{A(\bar{u}_r) + A'(\bar{u}_r)\bar{u}_r - \psi_0\}\alpha(t)\psi_0^{-1}w_t - \frac{N-1}{r} A'(\bar{u}_r)\alpha^2(t)w_r^2 + \bar{u}^p$$

or

$$\alpha'(t) \geq \{A(\bar{u}_r) - A(0) + A'(\bar{u}_r)\bar{u}_r\}\alpha(t)\psi_0^{-1}\frac{w_t}{w} - \frac{N-1}{r}A'(\bar{u}_r)\alpha^2(t)\frac{w_r^2}{w} + \frac{\bar{u}^p}{w}.$$

Here we put $\psi_0 = \psi(1) = A(0)$. Now let us apply mean value theorem to $A(s)$ on $[0, \bar{u}_r]$;

$$A(\bar{u}_r) - A(0) = \bar{u}_r A'(\theta\bar{u}_r), \qquad 0 < \theta < 1.$$

Since $w(r,t)$ is a solution of $w_t = \psi_0 \Delta w$, we have

$$\begin{aligned} w_t/w &= \frac{1}{2(t+t_0)}\left\{-N + \frac{r^2}{2(t+t_0)\psi_0}\right\}, \\ w_r/w &= \frac{-r}{2\psi_0(t+t_0)}. \end{aligned}$$

So

$$\begin{aligned} \alpha'(t) &\geq -\frac{N\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\theta\bar{u}_r) - \frac{\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\bar{u}_r) \\ &+ \frac{\alpha\bar{u}_r r^2}{4\psi_0^2(t+t_0)^2}\{A'(\theta\bar{u}_r) + A'(\bar{u}_r)\} + \frac{\bar{u}^p}{w}. \end{aligned}$$

Now let us define

$$\begin{aligned} f(t) &= -\frac{N\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\theta\bar{u}_r) - \frac{\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\bar{u}_r) \\ &+ \frac{\alpha\bar{u}_r r^2}{4\psi_0^2(t+t_0)^2}\{A'(\theta\bar{u}_r) + A'(\bar{u}_r)\} + \frac{\bar{u}^p}{w}. \end{aligned}$$

Then

$$\begin{aligned} f(t) &\leq \left|-\frac{N\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\theta\bar{u}_r)\right| + \left|-\frac{\alpha\bar{u}_r}{2\psi_0(t+t_0)}A'(\bar{u}_r)\right| \\ &+ \left|\frac{\alpha\bar{u}_r r^2}{4\psi_0^2(t+t_0)^2}\{A'(\theta\bar{u}_r) + A'(\bar{u}_r)\}\right| + \left|\frac{\bar{u}^p}{w}\right|, \\ &= \frac{N\alpha^2}{2\psi_0(t+t_0)}|w_r|.|A'(\theta\bar{u}_r)| + \frac{\alpha^2}{2\psi_0(t+t_0)}|w_r|.|A'(\bar{u}_r)| \\ &+ \frac{\alpha^2}{4\psi_0^2(t+t_0)^2}|r^2 w_r|\left\{|A'(\theta\bar{u}_r)| + |A'(\bar{u}_r)|\right\} + \alpha^p w^{p-1}. \end{aligned}$$

Now using $e^{-x} < x^{-1/2}$, we get

$$\begin{aligned} |w_r| &= w_0(t+t_0)^{-N/2}\frac{r}{2(t+t_0)\psi_0}.exp(-\frac{r^2}{4(t+t_0)\psi_0}) \\ &< w_0(t+t_0)^{-N/2}\frac{r}{2(t+t_0)\psi_0}.\frac{2\psi_0^{1/2}(t+t_0)^{1/2}}{r} \\ &= C_1(t+t_0)^{-\frac{1}{2}(N+1)} \end{aligned}$$

for some constant C_1. In a similar way, using $e^{-x^2} < x^{-3}$, we obtain

$$|r^2 w_r| < C_2(t+t_0)^{-\frac{1}{2}(N-1)}.$$

Let

$$A^*(t_0) := max\{|A'(s)|||s| \leq C_1(t+t_0)^{-\frac{1}{2}(N+1)}\}.$$

So we have

$$\begin{aligned} f(t) &\leq \frac{N\alpha^2}{2\psi_0(t+t_0)}C_1(t+t_0)^{-\frac{1}{2}(N+1)}A^*(t_0) \\ &+ \frac{\alpha^2}{2\psi_0(t+t_0)}C_1(t+t_0)^{-\frac{1}{2}(N+1)}A^*(t_0) \\ &+ \frac{\alpha^2}{4\psi_0^2(t+t_0)^2}2C_2(t+t_0)^{-\frac{1}{2}(N-1)}A^*(t_0) \\ &+ \alpha^p w_0^{p-1}(t+t_0)^{-(p-1)N/2}.exp(-\frac{r^2(p-1)}{4(t+t_0)\psi_0}) \end{aligned}$$

or

$$\begin{aligned} f(t) &\leq \frac{A^*(t_0)}{2\psi_0}\{(N+1)C_1 + C_2/\psi_0\}(t+t_0)^{-\frac{1}{2}(N+3)}\alpha^2(t) \\ &+ w_0^{p-1}\alpha^p(t)(t+t_0)^{-(p-1)N/2}. \end{aligned}$$

Then $\bar{u}$ will be a supersolution if $\alpha(t)$ solves

$$\begin{aligned} \alpha'(t) &= \frac{A^*(t_0)}{2\psi_0}\{(N+1)C_1 + C_2/\psi_0\}(t+t_0)^{-\frac{1}{2}(N+3)}\alpha^2(t) \\ &+ w_0^{p-1}\alpha^p(t)(t+t_0)^{-(p-1)N/2} \end{aligned} \quad (3.3)$$

and

$$0 < \alpha \leq 1. \tag{3.4}$$

Clearly α is increasing. To show that (3.3) and (3.4) has a solution, let

$$\epsilon = min\{(p-1)N/2, (N+3)/2\} - 1.$$

Then $\epsilon > 0$ and α satisfies

$$\alpha'(t) \leq \{A_1\alpha^2(t) + A_2\alpha^p(t)\}(t+t_0)^{-1-\epsilon} \tag{3.5}$$

for positive A_1, A_2 depending only on t_0, C_1, C_2, w_0, ψ_0. Now choose $\alpha(0) = \alpha_0 > 0$ but so small that

$$\int_{\alpha_0}^{1} \frac{d\sigma}{A_1\sigma^2 + A_2\sigma^p} > \epsilon^{-1}t_0^{-\epsilon}.$$

(This possible since $p \geq 1$.) Then from (3.5) we get

$$\int_0^t \frac{d\alpha(s)}{A_1\alpha^2(s) + A_2\alpha^p(s)} \leq \int_0^t (s+t_0)^{-1-\epsilon} ds$$

or

$$\int_{\alpha_0}^{\alpha(t)} \frac{d\sigma}{A_1\sigma^2 + A_2\sigma^p} \leq \epsilon^{-1}(t_0^{-\epsilon} - (t+t_0)^{-\epsilon}) < \epsilon^{-1}t_0^{-\epsilon}.$$

So we have

$$\int_{\alpha_0}^{\alpha(t)} \frac{d\sigma}{A_1\sigma^2 + A_2\sigma^p} < \epsilon^{-1}t_0^{-\epsilon} < \int_{\alpha_0}^{1} \frac{d\sigma}{A_1\sigma^2 + A_2\sigma^p}.$$

Thus, $\alpha(t) < 1$ for all $t > 0$. This completes the proof of the theorem.

CHAPTER 4

EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS FOR INITIAL- BOUNDARY VALUE PROBLEMS

4.1 Introduction

In this chapter we will discuss the long time behaviour of nonnegative solutions of

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + u^p \qquad \text{in } \; D \times (0,T), & (4.1)\\ u(r,\theta,t) &= 0 \qquad \text{on } \; \partial D \times (0,T) \; \text{ and at } \; r=\infty, & (4.2)\\ u(r,\theta,0) &= u_0(r,\theta) \qquad \text{in } \; D \times \{0\}, & (4.3) \end{aligned}$$

where $u_0 \geq 0$ is given, $p > 1$, [3]. Here $D = \{(r,\theta) | r > 0, \theta \in \Omega\}$ defines a cone in $\mathbb{R}^N$ where $\Omega \subset S^{N-1}$ be a connected submanifold of the unit sphere S^{N-1} in $\mathbb{R}^N$ with boundary $\partial\Omega$ and having positive $N-1$ dimensional measure.

Now let $\lambda = -\gamma_-$ where γ_- is the negative root of $\gamma(\gamma+N-2) = \omega_1$ and where ω_1 is the smallest Dirichlet eigenvalue of the Laplace-Beltrami operator on Ω, namely

$$\begin{aligned} \Delta_\theta \psi + \omega_1 \psi &= 0 \qquad \text{in} \quad \Omega,\\ \psi &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

It is known that we may take $\psi > 0$ on Ω. We shall assume ψ is normalized so that

$$\int_\Omega \psi(\theta) dS_\theta = 1.$$

We prove: If $1 < p < 1 + 2/(2+\lambda)$, there are no non-trivial global solutions of (4.1)-(4.3). If $p > 1 + 2/(2+\lambda)$, nontrivial global solutions of (4.1)-(4.3) do exist.

Definition 4.1.1 *(Quasiregularity) A solution of (4.1)-(4.3) is called quasiregular in* $Q_T := D \times (0,T)$ *if*

(i) $u \in C^2(Q_T) \bigcap C^0(\bar{Q_T} - D \times \{T\})$*;*

(ii) $\forall k > 0$,

$$\lim_{r\to\infty} e^{-kr} \int_\Omega |u(r,\theta,t)| dS_\theta = 0 \quad and \quad \lim_{r\to\infty} e^{-kr} \int_\Omega |u_r(r,\theta,t)| dS_\theta = 0.$$

4.2 Nonexistence of Global Solutions in Sectorial Domains:

To prove the nonexistence of global solutions of the equation (4.1)-(4.3), we need the following lemmas. [3]

Lemma 4.2.1 *Let* m, k, λ, ω *be real constants with* $k^2 + \lambda > 0$*. Let* $\varphi(r) = r^m e^{-kr}$*. If one of the conditions*

$$(\mathbf{A}) \qquad (k^2+\lambda)(m^2 + (N-2)m - \omega) \geq (m + \frac{1}{2}(N-1)^2)k^2$$

or

$$(\mathbf{B}) \qquad \begin{aligned} &(i) \quad (k^2+\lambda)(m^2 + (N-2)m - \omega) < (m + \frac{1}{2}(N-1)^2)k^2, \\ &(ii) \quad (m^2 + (N-2)m - \omega) > 0, \\ &(iii) \quad k(m + \frac{1}{2}(N-1)) < 0 \end{aligned}$$

holds, then

$$r^{N-1} \frac{d}{dr}(r^{N-1} \frac{d\varphi(r)}{dr}) \geq \omega r^{-2} \varphi(r) - \lambda \varphi(r);$$

that is,

$$\Delta\varphi(r) + \lambda\varphi(r) \geq \omega\varphi(r)r^{-2}$$

for all $r > 0$.

Proof. By a direct computation we find

$$\begin{aligned} & \frac{d}{dr}\{r^{N-1}\varphi(r)(\frac{m}{r} - k)\} = \frac{d}{dr}\{\varphi(r)(mr^{N-2} - kr^{N-1})\} \\ = \;& \varphi(r)\Big\{(m^2 + m(N-2))r^{N-3} - (k(N-1) + 2mk)r^{N-2} + k^2 r^{N-1}\Big\} \end{aligned}$$

and

$$r^{-(N-1)}\frac{d}{dr}\{r^{N-1}\varphi'(r)\} = \varphi(r)\Big\{(m^2+m(N-2))r^{-2} - (k(N-1)+2mk)r^{-1} + k^2\Big\}$$

which leads to

$$r^{N-1}\frac{d}{dr}(r^{N-1}\frac{d\varphi(r)}{dr}) - \omega r^{-2}\varphi(r) + \lambda\varphi(r) = r^{-2}\varphi(r)P(r)$$

where

$$P(r) = (\lambda + k^2)r^2 - (2mk + k(N-1))r + (m^2 + m(N-2) - \omega).$$

Now we claim that $P(r) > 0$, $\forall r > 0$; **A** implies that discriminant of $P(r)$ is negative and since $\lambda + k^2 > 0$, thus $P(r)$ has no real roots and $P(r) > 0$.

B(**i**) implies that discriminant of $P(r)$ is positive, **B**(**ii**) implies that $r_1.r_2 > 0$, **B**(**iii**) implies that $r_1 + r_2 < 0$. So we can conclude that $r_2 \leq r_1 < 0$ and $P(r) \leq 0$ on $[r_2, r_1]$ i.e. $P(r) > 0$ on outside of (r_2, r_1), since $\lambda + k^2 > 0$. But $r > 0$, therefore $P(r) > 0$, $\forall r > 0$.

Therefore

$$r^{N-1}\frac{d}{dr}(r^{N-1}\frac{d\varphi(r)}{dr}) - \omega r^{-2}\varphi(r) + \lambda\varphi(r) = r^{-2}\varphi(r)P(r) > 0.$$

Since

$$\begin{aligned}\Delta\varphi(r) &= \varphi_{rr}(r) + \frac{N-1}{r}\varphi_r(r) \\ &= r^{-(N-1)}\Big\{r^{N-1}\varphi_{rr}(r) + (N-1)r^{N-2}\varphi_r(r)\Big\} \\ &= r^{N-1}\frac{d}{dr}(r^{N-1}\frac{d\varphi(r)}{dr}),\end{aligned}$$

we get

$$\Delta\varphi(r) + \lambda\varphi(r) \geq \omega\varphi r^{-2}.$$

Lemma 4.2.2 *Let $G(t)$ be a nonnegative C^1 function defined on $[0, T)$ which satisfies*

$$G'(t) \geq G^p(t) - \lambda G(t) \quad \textit{for some} \quad \lambda \in \mathbb{R}.$$

If

(A) $G(0) > 0$ *and* $\lambda \leq 0$, *then*

$$T \leq G^{1-p}(0)/(p-1),$$

while if

(B) $\lambda > 0$ *and* $G(0) > \lambda^{1/(p-1)}$, *then*

$$T \leq \int_{G(0)}^{\infty} [\sigma^p - \lambda\sigma]^{-1} d\sigma$$

Proof.(A) Since $G(t)$ is nonnegative, for some $T > 0$

$$G'(t) \geq G^p(t) - \lambda G(t) \geq G^p(t),$$

$$\int_0^T dt \leq \int_0^T G^{-p} dG(t) \quad \text{and} \quad T \leq \frac{G^{1-p}(0)}{p-1}$$

(B). If $\lambda > 0$ and $G(0) > \lambda^{1/(p-1)}$ then,

$$1 \leq \frac{G'(t)}{G^p(t) - \lambda G(t)} \quad \Rightarrow \quad \int_0^T dt \leq \int_0^T \frac{dG(T)}{G^p(t) - \lambda G(t)}$$
$$\Rightarrow \quad T \leq \int_{G(0)}^{G(T)} \frac{d\sigma}{\sigma^p - \lambda\sigma} \leq \int_{G(0)}^{\infty} \frac{d\sigma}{\sigma^p - \lambda\sigma}$$

since $\sigma \geq G(0) > \lambda^{1/(p-1)}$, $\frac{1}{\sigma^p - \lambda\sigma} > 0$.

Theorem 4.2.3 *If*

$$1 < p < 1 + 2/(N + \gamma_+) = 1 + 2/(2 - \gamma_-)$$

then no almost regular solution of (4.1)-(4.3) with nontrivial, nonnegative inital data can exist for all time. [3], [10].

Proof. Let $u(r,\theta,t)$ be any quasiregular solution of (4.1)-(4.3) and define

$$\tilde{u}(r,t) := \int_\Omega \psi(\theta) u(r,\theta,t) dS_\theta \qquad \text{and} \qquad \tilde{u}_0(r) = \tilde{u}(r,0).$$

where ψ is normalized eigenfunction of the Laplace-Beltrami operator associated with ω_1.

Note that N dimensional Laplace operator can be written in polar coordinates as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta.$$

If we multiply (4.1) by $\psi(\theta)$ and integrate over Ω, we find

$$\tilde{u}_t(r,t) = \int_\Omega \psi(\theta) u_t(r,\theta,t) dS_\theta = \int_\Omega \psi(\theta) \Delta u(r,\theta,t) dS_\theta + \int_\Omega \psi(\theta) u^p(r,\theta,t) dS_\theta.$$

From Jensen's inequality we know that

$$\int_\Omega \psi(\theta) u^p(r,\theta,t) dS_\theta \geq \left\{ \frac{\int_\Omega \psi(\theta) u(r,\theta,t) dS_\theta}{\int_\Omega \psi(\theta) dS_\theta} \right\}^p \int_\Omega \psi(\theta) dS_\theta = \tilde{u}^p(r,\theta).$$

Using Green's identity and the fact that $\psi(\theta) = 0$ and $u(r, \theta, t) = 0$ on $\partial\Omega$ we have

$$\begin{aligned}
& \int_\Omega \psi(\theta)\Delta u(r,\theta,t)dS_\theta \\
= & \int_\Omega \psi(\theta)u_{rr}(r,\theta,t)dS_\theta + \frac{(N-1)}{r}\int_\Omega \psi(\theta)u_r(r,\theta,t)dS_\theta \\
+ & \frac{1}{r^2}\int_\Omega \psi(\theta)\Delta_\theta u(r,\theta,t)dS_\theta \\
= & \frac{\partial^2}{\partial r^2}\int_\Omega \psi(\theta)u(r,\theta,t)dS_\theta + \frac{(N-1)}{r}\frac{\partial}{\partial r}\int_\Omega \psi(\theta)u(r,\theta,t)dS_\theta \\
+ & \frac{1}{r^2}\int_\Omega u(r,\theta,t)\Delta_\theta \psi(\theta)dS_\theta,
\end{aligned}$$

or

$$\begin{aligned}
\int_\Omega \psi(\theta)\Delta u dS_\theta &= \frac{\partial^2}{\partial r^2}\tilde{u}(r,t) + \frac{(N-1)}{r}\frac{\partial}{\partial r}\tilde{u}(r,t) - \frac{\omega_1}{r^2}\int_\Omega \psi(\theta)u(r,\theta,t)dS_\theta \\
&= \Delta\tilde{u}(r,t) - \frac{\omega_1}{r^2}\tilde{u}(r,t).
\end{aligned}$$

So we have obtained

$$\tilde{u}_t(r,t) = \Delta\tilde{u}(r,t) + \omega_1\tilde{u}(r,t)/r^2 \geq \tilde{u}^p(r,t). \tag{4.4}$$

For $m > -(N-1)$ let us define

$$\Phi_0(r) = r^m e^{-kr}/C$$

where

$$C = \int_0^\infty r^{(m+N-1)}e^{-kr}dr = k^{-(m+N)}\Gamma(m+N).$$

If we multiply both sides of 4.4 by $\Phi_0(r)r^{N-1}$ and integrate on $[\varepsilon, R]$, we get

$$\frac{d}{dt}\int_\varepsilon^R \tilde{u}(r,t)\Phi_0(r)r^{N-1}dr - \int_\varepsilon^R \Delta\tilde{u}(r,t)\Phi_0(r)r^{N-1}dr$$

$$+\frac{\omega_1}{r^2}\int_\varepsilon^R \tilde{u}(r,t)\Phi_0(r)r^{N-1}dr \geq \int_\varepsilon^R \tilde{u}^p(r,t)\Phi_0(r)r^{N-1}dr.$$

By simple computations we find

$$\begin{aligned}
& \int_\varepsilon^R \Delta\tilde{u}(r,t)\Phi_0(r)r^{N-1}dr \\
= & \int_\varepsilon^R \Big\{\tilde{u}''(r,t) + \frac{(N-1)}{r}\tilde{u}'(r,t)\Big\}r^{N-1}\Phi_0(r)dr \\
= & \int_\varepsilon^R \Big\{r^{N-1}\tilde{u}'\Big\}'\Phi_0(r)dr \\
= & \Big\{\Phi_0(r)\tilde{u}'(r,t) - \Phi_0'(r)\tilde{u}(r,t)\Big\}r^{N-1}\ |_\varepsilon^R + \int_\varepsilon^R \tilde{u}(r,t)\Delta\Phi_0(r)r^{N-1}dr.
\end{aligned}$$

Therefore we deduce that

$$\frac{d}{dt}\int_\varepsilon^R \tilde{u}(r,t)\Phi_0(r)r^{N-1}dr - \int_\varepsilon^R \tilde{u}(r,t)\Big\{\Delta\Phi_0(r) - \frac{\omega_1}{r^2}\Phi_0(r)\Big\}r^{N-1}dr$$
$$+\Big\{\Phi_0'(r)\tilde{u}(r,t) - \Phi_0(r)\tilde{u}_r(r,t)\Big\}r^{N-1}\ |_\varepsilon^R \geq \int_\varepsilon^R \tilde{u}^p(r,t)\Phi_0(r)r^{N-1}dr.$$

Since u is quasiregular

$$\lim_{R\to\infty} e^{-kR}\int_\Omega |u(R,\theta,t)|dS_\theta = 0 \quad \text{and} \quad \lim_{R\to\infty} e^{-kR}\int_\Omega |u_r(R,\theta,t)|dS_\theta = 0$$

for all $k > 0$. Then

$$\lim_{R\to\infty} R^{N-1}\Big\{\Phi_0'(R)\tilde{u}(R,t) - \Phi_0(R)\tilde{u}_r(R,t)\Big\}$$

$$\begin{aligned}
= & \lim_{R\to\infty} C^{-1}R^{m+N-1}e^{-kR}(\frac{m}{R} - k)\int_\Omega \psi(\theta)u(R,\theta,t)dS_\theta \\
- & \lim_{R\to\infty} C^{-1}R^{m+N-1}e^{-kR}\int_\Omega \psi(\theta)u_r(R,\theta,t)dS_\theta
\end{aligned}$$

converges to 0 as $R \to \infty$ for any $m \in \mathbb{R}$

Let us assume that u and m are such that there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty$, $\varepsilon_n \to 0$ as $n \to \infty$ with the property

$$\lim_{n\to\infty} \varepsilon_n^{N-1+m}\Big\{m\tilde{u}(\varepsilon_n,t)/\varepsilon_n - \tilde{u}_r(\varepsilon_n,t)\Big\} = 0.$$

Now consider

$$
\begin{aligned}
& r^{N-1}\Big\{\tilde u(r,t)\Phi_0'(r) - \tilde u_r(r,t)\Phi_0(r)\Big\}\ |_{r=\varepsilon} \\
= \ & \varepsilon^{N-1}\Big\{\tilde u(\varepsilon,t)\Phi_0'(\varepsilon) - \tilde u_r(\varepsilon,t)\Phi_0(\varepsilon)\Big\} \\
= \ & \Phi_0(\varepsilon)\varepsilon^{N-1}\Big\{\tilde u(\varepsilon,t)(m/\varepsilon - k) - \tilde u_r(\varepsilon,t)\Big\}
\end{aligned}
$$

and substituting ε_n for ε we get

$$
\begin{aligned}
= \ & \Phi_0(\varepsilon_n)\varepsilon_n^{N-1}\Big\{\tilde u(\varepsilon_n,t)(m/\varepsilon_n - k) - \tilde u_r(\varepsilon_n,t)\Big\} \\
= \ & C^{-1}e^{-k\varepsilon_n}\varepsilon_n^{m+N-1}\Big\{\tilde u(\varepsilon_n,t)(m/\varepsilon_n - k) - \tilde u_r(\varepsilon_n,t)\Big\} \\
- \ & k\varepsilon_n^{N-1}\Phi_0(\varepsilon_n)\tilde u(\varepsilon_n,t).
\end{aligned}
$$

Thus

$$
\lim_{\varepsilon\to 0}\varepsilon^{N-1}\Big\{\Phi_0'(\varepsilon)\tilde u(\varepsilon,t) - \Phi_0(\varepsilon)\tilde u_r(\varepsilon,t)\Big\} = 0.
$$

So we deduce

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty \tilde u(r,t)\Phi_0(r)r^{N-1}dr \ & - \ \int_0^\infty \tilde u(r,t)\Big\{\Delta\Phi_0(r) - \frac{\omega_1}{r^2}\Phi_0(r)\Big\}r^{N-1}dr \\
& \geq \ \int_0^\infty \tilde u^p(r,t)\Phi_0(r)r^{N-1}dr.
\end{aligned}
$$

By use of Jensen's inequality

$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty \tilde u(r,t)\Phi_0(r)r^{N-1}dr \ & - \ \int_0^\infty \tilde u(r,t)\Big\{\Delta\Phi_0(r) - \frac{\omega_1}{r^2}\Phi_0(r)\Big\}r^{N-1}dr \\
& \geq \ \Big\{\int_0^\infty \tilde u(r,t)\Phi_0(r)r^{N-1}dr\Big\}^p. \qquad (4.5)
\end{aligned}
$$

Now let us define

$$
G(t) := \int_0^\infty \tilde u(r,t)\Phi_0(r)r^{N-1}dr \qquad (4.6)
$$

then (4.5) can be written as

$$G'(t) \geq G^p(t) + \int_0^\infty \tilde{u}(r,t)\Big\{\Delta\Phi_0(r) - \frac{\omega_1}{r^2}\Phi_0(r)\Big\}r^{N-1}dr. \tag{4.7}$$

Let $\lambda > 0$ (i.e. $k^2 + \lambda > 0$ for all k). In the view of $1 < p < 1 + 2/(N+\gamma_+)$, it is possible to find an m such that $m \in (\gamma_+, 2/(p-1) - N)$ then $m^2 + m(N-2) - \omega_1 > 0$. From Lemma 4.2.1, putting $\omega = \omega_1 > 0$, we get

$$\Delta\phi(r) + \lambda\phi(r) \geq \frac{\omega_1}{r^2}\phi(r)$$

where $\phi(r) = r^m e^{-kr}$. If we multiply both sides by $C^{-1} = k^{(m+N)}\Gamma^{-1}(m+N)$, then we get

$$\Delta\Phi_0(r) - \frac{\omega_1}{r^2}\Phi_0(r) \geq -\lambda\Phi_0(r).$$

Hence from (4.7), we have

$$G'(t) \geq G^p(t) - \lambda G(t), \tag{4.8}$$

and from (4.6)

$$\begin{aligned} G(0) &= \int_0^\infty \tilde{u}_0(r)\Phi_0(r)r^{N-1}dr \\ &= C^{-1}\int_0^\infty r^{m+N-1}\tilde{u}_0(r)dr \\ &= \Big\{k^{-(m+N)}\Gamma(m+N)\Big\}^{-1}\int_0^\infty r^{m+N-1}\tilde{u}_0(r)dr. \end{aligned} \tag{4.9}$$

Since we have chosen $m \in (\gamma_+, 2/(p-1) - N)$, we can find a $\beta := \lambda/k^2$ such that

$$\beta \geq \frac{m + \omega_1 + (N-2)^2/4}{m^2 + m(N-2) - \omega}.$$

Since $m + N < 2/(p-1)$ and we can therefore find a k sufficiently small satisfying

$$k^{-[2/(p-1)-m-N]}\int_0^\infty r^{m+N-1}\tilde{u}_0(r)dr > \Gamma(m+N)\beta^{1/(p-1)}. \tag{4.10}$$

From (4.9) and (4.10) we get

$$G(0) > \lambda^{1/(p-1)}. \tag{4.11}$$

So we can apply Lemma 4.2.2(B) to get

$$T \leq \int_{G(0)}^{\infty} \frac{d\sigma}{\sigma^p - \lambda\sigma} < \infty.$$

This completes the proof of theorem.

4.3 Existence of Global Solutions in Sectorial Domains

For the existence of the global solutions of the equation (4.1)-(4.3), we have following theorem of Levine and Meier [10].

Theorem 4.3.1 *If $p > 1 + 2/(N + \gamma_+)$, nontrivial global solutions of (4.1)-(4.3) exist. [10]*

Proof. The proof proceeds by the method of supersolutions. Let $\bar{u}(x,t) = \beta(t)w(x,t)$ be the supersolution of (4.1) -(4.3) with $w(x,t)$ is the positive solution of $u_t = \Delta u$ in Q_T. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{u}(x,t) &\geq \Delta\bar{u}(x,t) + \bar{u}^p(x,t), \\
\beta'(t)w(x,t) + \beta(t)\frac{\partial}{\partial t}w(x,t) &\geq \beta(t)\Delta w(x,t) + \beta^p(t)w^p(x,t), \\
\beta'(t)w(x,t) &\geq \beta^p(t)w^p(x,t), \\
\beta'(t) &\geq \beta^p(t)w^{p-1}(x,t).
\end{aligned}$$

Then $\bar{u}$ is a supersolution of (4.1)-(4.3) if

$$\beta'(t) = \beta^p(t) \parallel w(x,t) \parallel_{\infty}^{p-1}, \qquad 0 < t < T. \tag{4.12}$$

The solution of (4.12) with $\beta(0) = \beta_0 > 0$ will be global if

$$W_\infty := \int_0^\infty \| w(x,t) \|_\infty^{p-1} \, dt < \infty \tag{4.13}$$

and if

$$0 < \beta_0 < \{(p-1)W_\infty\}^{-1/(p-1)}.$$

Let

$$W_T := \int_0^T \| w(x,t) \|_\infty^{p-1} \, dt = \int_0^T \frac{d\beta(t)}{\beta^p(t)},$$

then

$$W_T = \frac{\beta^{1-p}(T) - \beta_0^{1-p}}{1-p},$$

or

$$0 < \beta(T) = \frac{1}{(\beta_0^{1-p} - (p-1)W_T)^{p-1}}.$$

So $\beta(t)$ exits globally if $\beta_0 - (p-1)W_\infty > 0$ that is

$$0 < \beta_0 < \{(p-1)W_\infty\}^{-1/(p-1)}.$$

Thus it remains to construct $w(x,t)$ such that $W_\infty < \infty$. To do this, we let $r = |x|$ and let $t_0 > 0$ be fixed. We define

$$\nu := \gamma_+ + \frac{1}{2}(N-2) = \left\{\omega_1 + (\frac{1}{2}(N-2))^2\right\}^{1/2}.$$

It is easy to show that

$$w(r,\theta,t) = (t+t_0)^{-1} r^{\frac{1}{2}(N-2)} I_\nu(r/2(t+t_0)) e^{-\frac{r^2+1}{4(t+t_0)}} \psi(\theta)$$

where I_ν denotes the Bessel's function of order ν [17].

Since $\psi(\theta)$ is bounded in $\bar{\Omega}$, in order to show that (4.13) for $p > 1 + 2/(N+\gamma_+)$, it suffices to show that

$$\limsup_{t\to\infty}(t+t_0)^{\frac{1}{2}(N+\gamma_+)}\{\sup_{r>0} W(r,t)\} < \infty$$

where

$$W(r,t) := (t+t_0)^{-1} r^{\frac{1}{2}(N-2)} I_\nu\left(\frac{r}{2(t+t_0)}\right) e^{-(r^2+1)/4(t+t_0)}.$$

Since

$$\limsup_{t\to\infty}(t+t_0)^{\frac{1}{2}(N+\gamma_+)}\left\{\sup_{r>0} W(r,t)\right\} = C < \infty,$$

we have

$$\limsup_{t\to\infty}(t+t_0)^{\frac{1}{2}(N+\gamma_+)(p-1)}\left\{\sup_{r>0} W(r,t)\right\}^{(p-1)} = C^{(p-1)}$$

or

$$\limsup_{t\to\infty}\frac{\left\{\sup_{r>0} W(r,t)\right\}^{(p-1)}}{(t+t_0)^{-\frac{1}{2}(N+\gamma_+)(p-1)}} = C^{(p-1)}$$

so

$$\int_0^\infty \left\{\sup_{r>0} W(r,t)\right\}^{p-1} dt \quad \text{is convergent because}$$

$$\int_0^\infty \frac{dt}{(t+t_0)^{\frac{1}{2}(N+\gamma_+)(p-1)}} < \infty$$

for $\frac{1}{2}(N+\gamma_+)(p-1) > 1$. This implies that

$$\begin{aligned} W_\infty &= \int_0^\infty \| w(x,t) \|_\infty^{p-1} dt \\ &= \int_0^\infty \left\{\sup_{r>0} W(r,t)\psi(\theta)\right\}^{p-1} dt \\ &= \psi(\theta)\int_0^\infty \left\{\sup_{r>0} W(r,t)\right\}^{p-1} dt \end{aligned}$$

is also convergent. So we get (4.13).

Now in order to verify (4.13), we will show that

$$\limsup_{t\to\infty}(t+t_0)^{\frac{1}{2}(N+\gamma_+)}\left\{\sup_{r>0} W(r,t)\right\} < \infty.$$

First of all $W(r,t)$ vanishes at $r = 0, \infty$ because of the

$$I_\nu(z) \approx \begin{cases} 2^{-\nu}/\Gamma(\nu+1).z^\nu, & \text{as } z \to 0^+ \\ (2\pi z)^{-1/2}e^z, & \text{as } z \to +\infty. \end{cases}$$

Thus a value $r_m(t)$ of r may be found such that $0 < r_m(t) < \infty$ and

$$W(r_m(t),t) = \sup_{r>0} W(r,t).$$

Let

$$\begin{aligned} \mathcal{W}(t) &= (t+t_0)^{\frac{1}{2}(N+\gamma_+)}W(r_m(t),t) \\ &= (t+t_0)^{\frac{1}{2}(N+\gamma_+)}(t+t_0)^{-1}r_m^{-\frac{1}{2}(N-2)}(t)I_\nu(\frac{r_m(t)}{2(t+t_0)})e^{-\frac{r_m^2(t)}{4(t+t_0)}} \\ &= (t+t_0)^{\frac{\gamma_+}{2}}(\frac{r_m}{t+t_0})^{-\frac{1}{2}(N-2)}I_\nu(\frac{r_m(t)}{2(t+t_0)})e^{-\frac{r_m^2(t)}{4(t+t_0)}}. \end{aligned}$$

Let

$$y_m(t) = \frac{1}{2}r_m(t)/(t+t_0).$$

Suppose on some sequence $\{t_k\}_{k=1}^\infty$, $\mathcal{W}(t_k) \to \infty$ as $t_k \to +\infty$. If, on some subsequence, $y_m(t_k) \to 0$, then

$$\begin{aligned} \mathcal{W}(t_k) &= (t_k+t_0)^{\frac{\gamma_+}{2}}(\frac{r_m}{t_k+t_0})^{-\frac{1}{2}(N-2)}I_\nu(\frac{r_m(t_k)}{2(t_k+t_0)})e^{-\frac{r_m^2(t_k)}{4(t_k+t_0)}} \\ &= C(t_k+t_0)^{\frac{\gamma_+}{2}}(2y_m(t_k))^{-\frac{1}{2}(N-2)}\frac{2^{-\nu}}{\Gamma(\nu+1)}y_m^\nu(t_k)e^{-\frac{1}{4(t_k+t_0)}} \\ &\quad e^{-y_m^2(t_k)(t_k+t_0)} \\ &= C(t_k+t_0)^{\frac{\gamma_+}{2}}y_m^{\gamma_+}(t_k)e^{-\frac{1}{4(t_k+t_0)}}.e^{-y_m^2(t_k)(t_k+t_0)}, \end{aligned}$$

since $e^{-\frac{1}{4(t_k+t_0)}}$ is bounded,

$$\mathcal{W}(t_k) = C\Big\{y_m^2(t_k)(t_k+t_0)\Big\}^{\gamma_+/2} e^{-y_m^2(t_k)(t_k+t_0)} = Cz^{\gamma_+}e^{-z^2}$$

where $z = y_m(t_k)(t_k+t_0)^{1/2}$. Since $z^{\gamma_+}e^{-z^2}$ is bounded on $[0,\infty)$, $\mathcal{W}(t_k) \not\to \infty$ on such a subsequence. If, on the other hand $y_m(t_k) \to \infty$ on some subsequence.

$$\begin{aligned}
\mathcal{W}(t_k) &= (t_k+t_0)^{\frac{\gamma_+}{2}}(\frac{r_m}{t_k+t_0})^{-\frac{1}{2}(N-2)} I_\nu(y_m(t_k)) e^{-\frac{r_m^2(t_k)}{4(t_k+t_0)}}, \\
&= C(t_k+t_0)^{\frac{\gamma_+}{2}}(2y_m(t_k))^{-\frac{1}{2}(N-2)}(2\pi y_m(t_k))^{-\frac{1}{2}} e^{y_m(t_k)} e^{-\frac{-1}{4(t_k+t_0)}} \\
&\quad e^{-y_m^2(t_k)(t_k+t_0)} \\
&= C\{y_m^2(t_k)(t_k+t_0)\}^{\frac{\gamma_+}{2}} e^{-(y_m(t_k)\sqrt{t_k+t_0}-\frac{1}{2\sqrt{t_k+t_0}})^2}, \\
&= Cz^{\gamma_+}e^{-(z-\frac{1}{2\sqrt{t_k+t_0}})^2}
\end{aligned}$$

where $z = y_m(t_k)\sqrt{t_k+t_0}$ and

$$z^{\gamma_+}e^{-(z-\frac{1}{2\sqrt{t_k+t_0}})^2} \to 0 \qquad \text{as} \qquad z \to \infty,$$

from which we again conclude that $\mathcal{W}(t_k) \not\to \infty$. So have proved that as $y_m(t_k) \to 0$, $\mathcal{W}(t_k) \not\to \infty$ and as $y_m(t_k) \to \infty$, $\mathcal{W}(t_k) \not\to \infty$. Therefore, if $\mathcal{W} \to +\infty$, we must have constants A and B such that

$$0 < A \le y_m(t_k) \le B < \infty.$$

However, in this case

$$\begin{aligned}
\mathcal{W}(t_k) &= (t_k+t_0)^{\frac{\gamma_+}{2}}(2y_m(t_k))^{-\frac{1}{2}(N-2)} I_\nu(y_m(t_k)) e^{-\frac{1}{4(t_k+t_0)}} e^{-y_m^2(t_k)(t_k+t_0)} \\
&\le C(t_k+t_0)^{\frac{\gamma_+}{2}} e^{-A^2(t_k+t_0)} < C_1 < \infty.
\end{aligned}$$

So we get

$$\mathcal{W}(t) = (t+t_0)^{\frac{1}{2}(N+\gamma_+)}\Big\{\sup_{r>0} W(r,t)\Big\} \not\to \infty \quad \text{as} \quad t \to \infty.$$

Therefore we have

$$\limsup_{t\to\infty}(t+t_0)^{\frac{1}{2}(N+\gamma_+)}\{\sup_{r>0} W(r,t)\} < \infty.$$

This establishes the theorem.

4.4 Nonexistence of Global Stationary Solutions

For nonexistence of global stationary solutions of the equation (4.1)-(4.3), we have the following theorem of Bandle and Levine [3]

Theorem 4.4.1 *If*

$$1 < p < 1 - \frac{2}{\gamma_-}$$

then, there are no positive stationary solution of (4.1)-(4.3), that is, there exists no $w > 0$ such that

$$\Delta w + w^p = 0 \quad in \quad D, \qquad w = 0 \quad on\ \partial D. \tag{4.14}$$

where $D = \{(r,\theta) \mid r > 0, \theta \in \Omega\}$ and $\Omega \subset S^{N-1}$ is the connected submanifold of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Proof. Let u solve $\Delta u + u^p = 0$ in D, $u \in C^0(\bar{D}) \cap C^2(D')$ for all $D' \subset D, D'$ bounded, and let $u = 0$ on ∂D. Define

$$\tilde{u}(r) := \int_\Omega \psi(\theta) u(r,\theta) dS_\theta.$$

where ψ is the normalized eigenfunction of the Dirichlet problem defined by

$$\begin{aligned} \Delta_\theta \psi + \omega_1 \psi &= 0 \ \text{ in } \ \Omega, \\ \psi &= 0 \ \text{ on } \ \partial\Omega, \end{aligned}$$

and

$$\int_\Omega \psi(\theta) dS_\theta = 1$$

in which ω_1 is the smallest Dirichlet eigenvalue of the Laplace-Beltrami operator.

We know that the Laplace operator in N dimensions can be written in polar coordinates as $\Delta = \Delta_r + \frac{1}{r^2}\Delta_\theta$ where

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r}.$$

Then

$$\begin{aligned} \Delta \tilde{u}(r) &= \Delta_r \tilde{u}(r) \\ &= \int_\Omega \psi(\theta) \Delta_r u(r,\theta) dS_\theta = \int_\Omega \psi(\theta) \Big\{ \Delta u(r,\theta) - \frac{1}{r^2}\Delta_\theta u(r,\theta) \Big\} dS_\theta \\ &= -\int_\Omega \psi(\theta) u^p(r,\theta) dS_\theta - \frac{1}{r^2}\int_\Omega \psi(\theta)\Delta_\theta u(r,\theta) dS_\theta. \end{aligned} \tag{4.15}$$

Using the Green's identity for the second integral in (4.15) we get,

$$\begin{aligned} \int_\Omega \psi(\theta)\Delta_\theta u(r,\theta) dS_\theta &= \int_\Omega u(r,\theta)\Delta_\theta \psi(\theta) dS_\theta - \int_{\partial\Omega} \Big\{ \psi(\theta)\frac{\partial u(r,\theta)}{\partial n} \\ &- u(r,\theta)\frac{\partial \psi(\theta)}{\partial n} \Big\} dS_\theta. \end{aligned}$$

So

$$\begin{aligned} \Delta \tilde{u}(r) &= \frac{\omega_1}{r^2}\int_\Omega u(r,\theta)\psi(\theta) dS_\theta - \int_\Omega \psi(\theta) u^p(r,\theta) dS_\theta \\ &= \frac{\omega_1}{r^2}\tilde{u}(r) - \int_\Omega \psi(\theta) u^p(r,\theta) dS_\theta. \end{aligned}$$

If we use the Jensen's inequality, we obtain

$$\begin{aligned}\int_\Omega \psi(\theta)u^p(r,\theta)dS_\theta &\geq \left\{\frac{\int_\Omega u(r,\theta)\psi(\theta)dS_\theta}{\int_\Omega \psi(\theta)dS_\theta}\right\}^p \int_\Omega \psi(\theta)dS_\theta \\ &= \left\{\int_\Omega u(r,\theta)\psi(\theta)dS_\theta\right\}^p \\ &= \tilde{u}^p(r)\end{aligned}$$

since u^p is the convex function of u for $p > 1$. Consequently,

$$\Delta\tilde{u}(r) - (\frac{\omega_1}{r^2})\tilde{u}(r) + \tilde{u}^p(r) \leq 0. \tag{4.16}$$

Now let λ be one of γ_+, γ_- and $\beta = N - 1 + 2\lambda$. Then (4.16) implies that

$$(r^\beta(r^{-\lambda}\tilde{u}(r))')' + r^{\beta-\lambda}\tilde{u}^p(r) \leq 0.$$

Let $l(r) = r^{-\lambda}\tilde{u}(r)$, then

$$(r^\beta l'(r))' + r^{\beta-\lambda}\tilde{u}^p(r) \leq 0. \tag{4.17}$$

for $0 < \rho < r < \infty$. Integrating twice the inequality (4.17) we find

$$r^\beta l'(r) - \rho^\beta l'(\rho) + \int_\rho^r s^{\beta-\lambda}\tilde{u}^p(s)ds \leq 0 \tag{4.18}$$

or

$$l(r_1) - l(r_2) + \int_{r_2}^{r_1} \xi^{-\beta}d\xi \int_\rho^\xi s^{\beta-\lambda}\tilde{u}^p(s)ds \leq \frac{r_1^{1-\beta} - r_2^{1-\beta}}{1-\beta}\rho^\beta l'(\rho).$$

To complete the proof of the theorem we will need several Lemmas.

Lemma 4.4.2 *Let $\lambda = \gamma_+$. Then*

$$\lim_{\rho\to 0}\rho^\beta l'(\rho) = 0.$$

Proof. The function $h(r,\theta) = r^\lambda \psi(\theta)$ is harmonic. Since

$$\begin{aligned}
\Delta h(r,\theta) &= \frac{\partial^2}{\partial r^2} h(r,\theta) + \frac{(N-1)}{r}\frac{\partial}{\partial r} h(r,\theta) + \frac{1}{r^2}\Delta_\theta h(r,\theta) \\
&= \lambda(\lambda-1) r^{\lambda-2}\psi(\theta) + \lambda(N-1) r^{\lambda-2}\psi(\theta) - \omega_1 r^{\lambda-2}\psi(\theta) \\
&= r^{\lambda-2}\psi(\theta)(\lambda(\lambda+N-2)-\omega_1) \\
&= r^{-2} h(r,\theta)(\gamma_+(\gamma_+ + N-2)-\omega_1) \\
&= r^{-2} h(r,\theta)(\omega_1-\omega_1) \\
&= 0.
\end{aligned}$$

Therefore, if we set

$$D_\varepsilon = \{(r,\theta) \mid \varepsilon < r < 1, \theta \in \Omega\}$$

and since $u^p + \Delta u = 0$, we find

$$0 = \int_{D_\varepsilon} h\{u^p + \Delta u\} dx = \int_{D_\varepsilon} h \Delta u dx + \int_{D_\varepsilon} h u^p dx. \tag{4.19}$$

Using the Green's identity for the first integral, we get

$$\int_{D_\varepsilon} h\Delta u dx = \int_{D_\varepsilon} u \Delta h dx + \oint_{\partial D_\varepsilon} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds.$$

Since $\Delta h = 0$, (4.19) gives

$$0 = \oint_{\partial D_\varepsilon} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds + \int_{D_\varepsilon} h u^p dx. \tag{4.20}$$

Let $\Gamma_a = \partial D_\varepsilon \cap \{r = a\}$. Then

$$\oint_{\partial D_\varepsilon} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds = \oint_{\Gamma_1} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds + \oint_{\Gamma_\varepsilon} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds$$

on Γ_1; $n = \frac{r}{|r|} = r$, since $|r| = 1$. But

$$\oint_{\Gamma_1} \left\{ h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n} \right\} ds = \oint_{\Gamma_1} \left\{ h\frac{\partial u}{\partial r} - u\frac{\partial h}{\partial r} \right\} ds = C \tag{4.21}$$

where C is a constant and Γ_ε; $n = -\frac{r}{|r|} = -r$

$$\begin{aligned}
\oint_{\Gamma_\varepsilon}\Big\{h\frac{\partial u}{\partial n} - u\frac{\partial h}{\partial n}\Big\}ds &= \int_{\Gamma_\varepsilon}\Big\{r^\lambda\psi(\theta)\frac{\partial u(r,\theta)}{\partial n} - \lambda u(r,\theta)\psi(\theta)r^{\lambda-1}\frac{\partial r}{\partial n}\Big\}ds \\
&= -\int_\Omega\Big\{\varepsilon^\lambda\psi(\theta)u_\varepsilon(\varepsilon,\theta) - \lambda\varepsilon^{\lambda-1}u(\varepsilon,\theta)\psi(\theta)\Big\}\varepsilon^{N-1}dS_\theta \\
&= -\varepsilon^{\lambda+N-1}\frac{\partial}{\partial\varepsilon}\Big\{\int_\Omega\psi(\theta)u(\varepsilon,\theta)dS_\theta\Big\} \\
&+ \lambda\varepsilon^{\lambda+N-2}\int_\Omega\psi(\theta)u(\varepsilon,\theta)dS_\theta \\
&= -\varepsilon^{\lambda+N-1}\tilde{u}'(\varepsilon) + \lambda\varepsilon^{\lambda+N-2}\tilde{u}(\varepsilon) \\
&= -\varepsilon^{\lambda+N-1}(\tilde{u}'(\varepsilon) - \lambda\tilde{u}(\varepsilon)/\varepsilon). \qquad (4.22)
\end{aligned}$$

Substituting (4.21) and (4.22) in (4.20) we find

$$\varepsilon^{\lambda+N-1}\Big\{\tilde{u}'(\varepsilon) - \lambda\tilde{u}(\varepsilon)/\varepsilon\Big\} = \int_{D_\varepsilon} hu^p dx + C.$$

Since $\tilde{u}(r) = r^\lambda l(r)$ we have

$$\begin{aligned}
\varepsilon^{\lambda+N-1}\{\tilde{u}'(\varepsilon) - \lambda\tilde{u}(\varepsilon)/\varepsilon\} &= \varepsilon^{\lambda+N-1}\Big\{\lambda\varepsilon^{\lambda-1}l(\varepsilon) + \varepsilon^\lambda l'(\varepsilon) - \lambda\varepsilon^{\lambda-1}l(\varepsilon)\Big\}, \\
&= \varepsilon^{2\lambda+N-1}l'(\varepsilon), \\
&= \varepsilon^\beta l'(\varepsilon),
\end{aligned}$$

so

$$\varepsilon^\beta l'(\varepsilon) = \int_{D_\varepsilon} hu^p dx + C.$$

Since C is independent of ε and the integral is a decreasing function of ε, $\lim_{\varepsilon\to 0}\varepsilon^\beta l'(\varepsilon)$ exits. Suppose this limit is $\kappa \neq 0$. Then for all $\delta \in (0, |\kappa|)$, there is ρ_0 such that

$$|\rho^\beta l'(\rho) - \kappa| \leq \delta$$

that is

$$\frac{\kappa - \delta}{\rho^\beta} \leq l'(\rho) \leq \frac{\kappa + \delta}{\rho^\beta}$$

if $\rho \le \rho_0$. Integrating this over $[\varepsilon, \rho_0]$, we find

$$(\kappa-\rho)\int_{\varepsilon}^{\rho_0}\rho^{-\beta}d\rho \le \int_{\varepsilon}^{\rho_0} l'(\rho)d\rho \le (\kappa+\rho)\int_{\varepsilon}^{\rho_0}\rho^{-\beta}d\rho$$

$$\Rightarrow \qquad \frac{\kappa-\delta}{\beta-1}[\varepsilon^{1-\beta}-\rho_0^{1-\beta}] \le l(\rho_0)-l(\varepsilon) \le \frac{\kappa+\delta}{\beta-1}[\varepsilon^{1-\beta}-\rho_0^{1-\beta}].$$

For $\lambda = \frac{1}{2}(\beta - N + 1)$,

$$\lim_{r\to 0} r^{\lambda} l(r) = \lim_{r\to 0}\tilde{u}(r) = \tilde{u}(0) = 0.$$

If we multiply the last inequality through by ε^{λ}, we get

$$\frac{\kappa-\delta}{\beta-1}[\varepsilon^{2-N-\lambda}-\varepsilon^{\lambda}\rho_0^{1-\beta}] \le \varepsilon^{\lambda}l(\rho_0)-\varepsilon^{\lambda}l(\varepsilon) \le \frac{\kappa+\delta}{\beta-1}[\varepsilon^{2-N-\lambda}-\varepsilon^{\lambda}\rho_0^{1-\beta}].$$

Since $2-N-\lambda = 2-N-\gamma_+ < 2-N \le 2-2 = 0$, we have contradiction if $\varepsilon \to 0^+$ unless $\kappa = 0$.

Now if we take limit of (4.18) as $\rho \to 0$ we get

$$r^{\beta}l'(r) + \int_0^r s^{N+\lambda-1}\tilde{u}^p(s)ds \le 0. \qquad (4.23)$$

Lemma 4.4.3 *Let $\lambda = \gamma_+$. Then $l(r)$ decreasing and*

$$\lim_{r\to\infty} l(r) = 0.$$

Proof. From (4.23) it is trivial that $l(r)$ is decreasing. Let $l_0 = \lim_{r\to\infty} l(r)$ and assume $l_0 \neq 0$. Then integrating (4.23) with reepect to r, we have

$$\begin{aligned} \int_{\rho}^{r} l'(\xi)d\xi + \int_{\rho}^{r}\xi^{-\beta}d\xi\int_0^{\xi}s^{N+\lambda-1}\tilde{u}^p(s)ds &\le 0 \\ l(r)-l(\rho) + \int_{\rho}^{r}\xi^{-\beta}d\xi\int_0^{\xi}s^{N-1+\lambda(p+1)}l^p(s)ds &\le 0. \qquad (4.24) \end{aligned}$$

Taking limit of the inequality (4.24) as $r \to \infty$, we get

$$\int_\rho^\infty \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} l^p(s) ds \leq l(\rho) - \lim_{r\to\infty} l(r) = l(\rho) - l_0.$$

or

$$\int_\rho^\infty \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} l^p(s) ds < \infty. \tag{4.25}$$

Now we claim that $l_0 = 0$. (4.25) implies that

$$l_0^p \int_\rho^\infty \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} ds < \infty$$

$$(N + \lambda(p+1))^{-1} l_0^p \int_\rho^\infty \xi^{N+\lambda(p+1)-\beta} d\xi < \infty$$

$$(N + \lambda(p+1))^{-1} l_0^p \int_\rho^\infty \xi^{1+\lambda(p-1)} d\xi < \infty, \tag{4.26}$$

since $(N + \lambda(p+1))^{-1} > 0$ and $1 + \lambda(p-1) > 0$. (4.26) gives the result that $l_0 = 0$.

Lemma 4.4.4 *Let* $\lambda = \gamma_+$. *Then*

$$\lim_{r\to\infty} r^{(2+(p-1)\lambda))/p} l(r) = 0.$$

Proof. From (4.24) we have

$$\int_r^{2r} \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} l^p(s) ds \leq l(r) - l(2r)$$

which yields

$$\lim_{r\to\infty} \int_r^{2r} \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} l^p(s) ds = 0.$$

On the other hand by monotonicity of l

$$l^p(2r) \int_r^{2r} \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} ds \leq \int_r^{2r} \xi^{-\beta} d\xi \int_0^\xi s^{N-1+\lambda(p+1)} l^p(s) ds.$$

Thus

$$\begin{aligned}
&\lim_{r\to\infty} l^p(2r)\int_r^{2r}\frac{\xi^{1+\lambda(p-1)}}{1+\lambda(p-1)}d\xi \\
=\ &\frac{1}{\{1+\lambda(p-1)\}\{2+\lambda(p-1)\}}\lim_{r\to\infty} l^p(2r)(2r)^{2+\lambda(p-1)}\{1-(\frac{1}{2})^{2+\lambda(p-1)}\}.
\end{aligned}$$

So

$$\lim_{r\to\infty} l^p(2r)r^{2+\lambda(p-1)} = 0$$

which completes the proof of the Lemma. The above result also yields

Corollary 4.4.5 *Let* $\lambda = \gamma_+$. *Then*

$$\lim_{r\to\infty} r^\nu l(r) = 0$$

for all $\nu < (2-(p-1)\lambda)/(p-1)$.

Proof. Let $a = 2+(p-1)\lambda$. Take $\rho = r/2$ in (4.24). Then

$$\begin{aligned}
l(r/2)-l(r) &\geq \int_{r/2}^r \xi^{-\beta}d\xi\int_0^\xi s^{N-1+\lambda(p+1)}l^p(s)ds, \\
&\geq l^p(r)\int_{r/2}^r \xi^{-\beta}d\xi\int_0^\xi s^{N-1+\lambda(p+1)}ds, \\
&= (N+(p+1)\lambda)^{-1}l^p(r)\int_{r/2}^r \xi^{a-1}d\xi, \\
&\geq (N+(p+1)\lambda)^{-1}(1-\frac{1}{2^a})l^p(r)r^a.
\end{aligned}$$

So

$$C\lim_{r\to\infty} l^p(r)r^a \leq \lim_{r\to\infty}\{l(r/2)-l(r)\} = 0 \quad \text{which gives} \quad \lim_{r\to\infty} l^p(r)r^a = 0.$$

Since

$$\begin{aligned}
r^{a/p+a}l^p(r) &= Cr^{a/p}(l(r/2)-l(r)) \\
&\leq Cr^{a/p}l(r/2) \\
&= C2^{a/p}(r/2)^{a/p}l(r/2),
\end{aligned}$$

from Lemma 4.4.4 we get

$$\lim_{r\to\infty}(r/2)^{a/p}l(r/2)=0.$$

Therefore we see that

$$\lim_{r\to\infty} r^{\frac{a}{p}+a}l^p(r)=0 \quad \text{implies} \quad \lim_{r\to\infty} r^{\frac{a}{p^2}+\frac{a}{p}}l(r)=0.$$

Repeating this argument yields

$$\lim_{r\to\infty} r^{a(\frac{1}{p}+\frac{1}{p^2}+\dots)}l(r)=\lim_{r\to\infty} r^{\frac{a}{p-1}}l(r)=0$$

since $\nu<\frac{2+(p-1)\lambda}{p-1}=\frac{a}{p-1}$. This implies that

$$\lim_{r\to\infty} r^{\nu}l(r)=0, \qquad \forall \nu<\frac{2+(p-1)\lambda}{p-1}.$$

Lemma 4.4.6 *Let $\lambda=\gamma_+$. There exits $c>0$ such that*

$$l(r)r^{N-2+2\lambda}\geq c \quad \textit{if} \quad r\geq 1.$$

Proof. From (4.23), we find

$$r^{\beta}l'(r)+\int_0^1 s^{N-1+\lambda}\tilde{u}^p(s)ds=r^{\beta}l'(r)+c_0$$

where $c_0=\int_0^1 s^{N-1+\lambda}\tilde{u}^p(s)ds$. So for $1\leq\rho\leq r$ we get

$$\begin{aligned} l'(r)+c_0r^{-\beta} &\geq \int_\rho^r l'(s)ds+c_0\int_\rho^r s^{-\beta}ds \\ &= l(r)-l(\rho)+c_0\Big\{\frac{r^{1-\beta}-\rho^{1-\beta}}{1-\beta}\Big\}. \end{aligned}$$

Letting $r \to \infty$, we have

$$l(\rho) \geq \frac{c_0}{\beta - 1}\rho^{1-\beta}, \qquad (\rho \geq 1)$$

or

$$\rho^{\beta-1} l(\rho) \geq C, \qquad (\rho \geq 1).$$

Substituting $\beta = N - 1 + 2\lambda$ and $\rho = r$ we get,

$$r^{N-2+2\lambda} l(r) \geq C. \qquad (r \geq 1)$$

To complete the proof of the theorem, we will use Corollary 4.4.5 and Lemma 4.4.6 with $\lambda = \gamma_+$:

$$\frac{(2 - \lambda + p\lambda)}{p - 1} \leq N - 2 + 2\lambda$$

or

$$p \geq 1 + \frac{2}{N - 2 + \lambda} = 1 + \frac{2}{N - 2 + \gamma_+} = 1 - \frac{2}{\gamma_-}$$

since $\gamma_+ + \gamma_- = 2 - N$. Thus if

$$1 < p < 1 - \frac{2}{\gamma_-},$$

we have shown that no positive stationary solution exits.

Theorem 4.4.7 *[3] If a stationary solution exits then $\tilde{u}(r)r^v \to 0$ as $r \to 0$ for all $v < 2/(p-1)$.*

Proof. If a stationary solution exits then

$$p \geq 1 - \frac{2}{\gamma_-} \quad \text{that is} \quad 0 < \frac{2}{p-1} \leq -\gamma_-.$$

This implies that

$$0 \leq |\tilde{u}(r)r^v| \leq \tilde{u}(r)r^{\frac{2}{p-1}} \leq \tilde{u}(r)r^{-\gamma_-} \to 0 \qquad \text{as} \qquad r \to 0,$$

since $\lim_{r\to 0} \tilde{u}(r) = \tilde{u}(0) = 0$ and $\lim_{r\to 0} r^{-\gamma_-} = 0$. Therefore we get,

$$\lim_{r\to 0} \tilde{u}(r)r^v = 0, \qquad \forall v < 2/(p-1).$$

4.5 Initial Boundary Value Problem on A Bounded Domain:

Let us consider existence and nonexistence of global solutions of the problem

$$u_t = \Delta u + h(t)u^p, \qquad \text{in } D_T := D \times (0,T] \tag{4.27}$$

$$u(x,0) = u_0(x) \geq 0, \qquad \text{in } D \tag{4.28}$$

$$u(x,t) = 0 \qquad \text{on } \partial D \times (0,T] \tag{4.29}$$

where D is a bounded domain in $\mathbb{R}^N$ with sectionaly smooth boundary. u_0 is bounded and $p > 1$. The function $h(t)$ has the properties

$(h1)$ $\quad h \in C[0,\infty), \quad h \geq 0;$

$(h2)$ $\quad \alpha_0 e^{\beta t} \leq h(t) \leq \alpha_1 e^{\beta t}$ for sufficiently large t,

where α_0, $\alpha_1 > 0$ and $\beta > 0$ are constants. Now let w be the solution of the equation

$$u_t = \Delta u \qquad \text{in } D_T \tag{4.30}$$

$$u(x,0) = u_0(x) \qquad \text{in } D \tag{4.31}$$

$$u(x,t) = 0 \qquad \text{on } \partial D \times (0,T]. \tag{4.32}$$

Then we have the following result of Meier [12].

Theorem 4.5.1 *Assume $h(t)$ has the property $(h1)$ but not necessarily $(h2)$.*

(i) *If there is a solution $w \not\equiv 0$ of (4.30) -(4.32) such that*

$$\int_0^\infty h(t) \parallel w(.,t) \parallel^{p-1} dt < \infty,$$

then there are global positive solutions u of (4.27)- (4.29) with

$$\lim_{t\to\infty} \parallel u(.,t) \parallel = 0$$

(ii) *If*

$$\limsup_{t\to\infty} \parallel w(.,t) \parallel^{p-1} \int_0^t h(s)ds = \infty$$

for all solutions $w \not\equiv 0$ of (4.30)-(4.32), then every nontrivial solution of (4.27)-(4.29) blows up in finite time.

Proof.(i) Let $a(t)$ be the solution of

$$a'(t) = h(t) \parallel w(.,t) \parallel^{p-1} a^p(t), \qquad a(0) = a_0 > 0.$$

Then $\bar{u}(x,t) := a(t)w(x,t)$ is a supersolution of for (4.27)-(4.29) with $\bar{u}(x,0) = a_0 w(x,0) = u_0(x)$;

$$\begin{aligned} & \bar{u}_t - \Delta\bar{u} - h(t)\bar{u} \\ = \;& a'(t)w(x,t) + a(t)w_t(x,t) - \Delta a(t)w(x,t) - h(t)a^p(t)w^p(x,t), \\ = \;& h(t) \parallel w(.,t) \parallel^{p-1} a^p(t)w(x,t) + a(t)\Delta w(x,t) - a(t)\Delta w(x,t) \\ - \;& h(t)a^p(t)w^p(x,t), \\ = \;& h(t)a^p(t)w(x,t)\Big\{\parallel w(.,t) \parallel^{p-1} - w^{p-1}(x,t)\Big\} \geq 0. \end{aligned}$$

We may assume that $\lim_{t\to\infty} \parallel w(.,t) \parallel = 0$ holds, since in every domain D there are solutions of (4.30) -(4.32) with that property. Thus it is enough to show that choice of a_0 ensures that $a(t)$ exists all over $\mathbb{R}_0^+$ and is uniformly bounded there. Now let us find $a(t)$; since $a'(t) = h(t) \parallel w(.,t) \parallel^{p-1} a^p(t)$,

$$
\begin{aligned}
\frac{a'(t)}{a^p(t)} &= h(t) \parallel w(x,t) \parallel^{p-1} \\
\int_0^t a^{-p}(s)da(s) &= \int_0^t h(s) \parallel w(.,s) \parallel^{p-1} ds \\
\int_{a_0}^{a(t)} \sigma^{-p} d\sigma &= \int_0^t h(s) \parallel w(.,s) \parallel^{p-1} ds
\end{aligned}
$$

and

$$
a(t) = \left\{ a_0^{-(p-1)} - (p-1) \int_0^t h(s) \parallel w(.,s) \parallel^{p-1} ds \right\}^{-1/(p-1)}.
$$

This means that we can choose a_0 so that $a(t)$ exists over all $\mathbb{R}_0^+$. Since we have

$$
\int_0^\infty h(s) \parallel w(.,s) \parallel^{p-1} ds = M < \infty,
$$

$$
\begin{aligned}
\lim_{t\to\infty} \parallel u(.,t) \parallel &\leq \lim_{t\to\infty} \parallel \bar{u}(.,t) \parallel \\
&= \lim_{t\to\infty} a(t) \parallel w(.,t) \parallel \\
&= \{a_0^{-(p-1)} - (p-1)M\}^{-1/(p-1)} \lim_{t\to\infty} \parallel w(.,t) \parallel \\
&= 0,
\end{aligned}
$$

therefore

$$
\lim_{t\to\infty} \parallel u(.,t) \parallel = 0.
$$

(ii) Define $z(v;w)$ to be the solution of

$$
\frac{dz}{dv} = z^p, \qquad z(0;w) = w. \tag{4.33}
$$

For arbitrary initial conditions let w be the solution of (4.30)-(4.32) and $v := \int_0^t h(s)ds$. Then $\underline{u}(x,t) := z(v(t); w(x,t))$ is a subsolution for (4.27) -(4.29) that blows up in finite time; for the proof of this let us look at

$$\begin{aligned}
\underline{u}_t - \Delta \underline{u} - h(t)\underline{u}^p &= \frac{dz}{dv}\frac{dv}{dt} + \frac{dz}{dw}\frac{dw}{dt} - \Delta z - h(t)z^p \\
&= z^p h(t) + \frac{dz}{dw}\Delta w - \Delta z - h(t)z^p \\
&= z_w \Delta w - \Delta z \\
&= z_w \Delta w - z_{ww}\sum_{i=1}^{N} |w_{x_i}|^2 - z_w \Delta w \\
&= -z_{ww}|\nabla w|^2 \le 0.
\end{aligned}$$

So we have $\underline{u}(x,t) = z(v(t), w(x,t))$ is the subsolution of (4.27)-(4.29) that is

$$\underline{u}(x,t) \le u(x,t).$$

If we solve the differential equation (4.33), we get

$$z(x,t) = \{w^{1-p}(x,t) - (p-1)v(t)\}^{1/(1-p)}.$$

So $z(v,w)$ blows up if and only if $w^{1-p}(x,T) = (p-1)v(T)$ for some $T < \infty$. This implies that

$$\| \, w(.,T) \, \|^{p-1} \int_0^T h(s)ds = \frac{1}{p-1}. \tag{4.34}$$

(4.34) is possible because

$$\limsup_{t\to\infty} \| \, w(.,t) \, \|^{p-1} \int_0^t h(s)ds = \infty.$$

Example: Let us consider the differential equation

$$u_t = \Delta u + ce^{\beta t}u^p, \qquad \text{in } D_T := D \times (0,T] \tag{4.35}$$

$$u(x,0) = u_0(x) \ge 0, \qquad \text{in } D \tag{4.36}$$

$$u(x,t) = 0 \qquad \text{on } \partial D \times (0,T]. \tag{4.37}$$

clearly the function $ce^{\beta t}$ satisfies the conditions $(h1)$ and $(h2)$ for sufficiently large t and for $\beta > 0$. Then we have the following theorem [12].

Theorem 4.5.2 *Assume D is bounded subset of $\mathbb{R}^N$ and $h(t)$ has the properties $(h1)$ and $(h2)$. Then*

(**a**) *If $p > p^*_\beta$, then there is a global solution $u \not\equiv 0$ of (4.35)-(4.37), whose supremum norm $\| u(.,t) \|$ is finite for any $t \geq 0$.*

(**b**) *If $1 < p < p^*_\beta$, then any nontrivial solution of (4.35)-(4.37) blows up in finite time, i.e. there exits $T_\infty < \infty$ such that $\lim_{t\to T_\infty} \| u(.,t) \| = \infty$ hold for $p^*_\beta = 1 + \beta/\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian in D.*

Proof. If $\phi(x)$ is a positive eigenfunction corresponding to the first eigenvalue λ_1 of the Laplacian.

Then $w(x,t) := e^{-\lambda_1 t}\phi_1(x)$ satisfies

$$w_t(x,t) - \Delta w(x,t) = -\lambda_1 e^{-\lambda_1 t}\phi_1(x) - e^{\lambda_1 t}\Delta\phi_1(x) = 0.$$

Therefore (**a**) follows from Theorem 4.5.1(i).

Now let us prove (**b**). For this it is enough to show that $L_\infty = \infty$ for $1 < p < p^*_\beta$ (i.e. $\beta - \lambda_1(p-1) > 0$) where

$$L_\infty = \limsup_{t\to\infty} \| w(x,t) \|^{p-1} \int_0^t ce^{\beta t} dt.$$

So

$$\begin{aligned} L_\infty &= c\beta^{-1} \limsup_{t\to\infty} \{\sup_{x\in D} \phi_1(x)\}^{p-1} e^{-\lambda_1(p-1)t}(e^{\beta t} - 1) \\ &= c\beta^{-1} \{\sup_{x\in D} \phi_1(x)\}^{p-1} \limsup_{t\to\infty} \Big\{ e^{[\beta-\lambda_1(p-1)]t} - e^{-\lambda_1(p-1)t} \Big\}. \end{aligned}$$

We can say that $\varepsilon = \beta - \lambda_1(p-1)$ for some $\varepsilon > 0$ and being the first Dirichlet eigenvalue, λ_1 is a positive number i. e. $\lambda_1(p-1) > 0$. So we get

$$L_\infty = c\beta^{-1}\{\sup_{x\in D}\phi_1(x)\}^{p-1}\limsup_{t\to\infty}\left\{e^{\varepsilon t} - e^{-\lambda_1(p-1)t}\right\} = \infty.$$

Therefore (**b**) follows from Theorem 4.5.1(ii).

REFERENCES

[1] J. Aguirre and M. Escobedo: *On the blow up of solutions of a convective reaction diffusion equation*, Proc. Roy. Soc. Edinburgh Sec. A, pp. 433-460, 123, 1993.

[2] C. Bandle and H. A. Levine: *Fujita type phenomena for reaction-diffusion equations with convection like terms*, Differential Integral Equations, pp. 1169-1193, 5-6, 1994.

[3] C. Bandle and H. A. Levine: *On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains*, Trans. Amer. Math. Soc., pp. 595-624, 655, 1989.

[4] H. Fujita: *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A Math. pp. 105-113, 16, 1966.

[5] V. A. Galaktionov, H. A. Levine: *On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary*, Israel J. of Math., pp. 125-146, 94, 1997.

[6] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, Cambridge University, p. 151, 1964.

[7] C. S. Jiang and C. H. Xie: *Blow up of solutions to quasilinear parabolic*

equations on a semi-infinite space, Acta Math. Appl. Sinica, pp. 84-91, 1998.

[8] H. A. Levine: *The role of critical exponents in blowup theorems*, SIAM Review, pp. 262-288, 32, 1990.

[9] H. A. Levine, G. M. Lieberman, and P. Meier: *On critical exponents for some quasilinear parabolic equations*, Math. Meth. Appl. Sci., pp. 429-438, 12, 1990.

[10] H. A. Levine and P. Meier: *The value of the critical exponent for reaction-diffusion equations in cones*, Arch. Rational Mech. Anal. pp. 73-80, 109, 1990.

[11] J. L. Lions, Quelques methodés de résolution des problémes aux limites non linéaries, Gauthier-Villars, 1969.

[12] P. Meier: *On the critical exponent for reaction-diffusion equations*, Arch. Rational Mech. Anal. pp. 63-71, 109, 1990.

[13] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum, 1992.

[14] Y. Qui: *The critical exponents of parabolic equations and blow up in* $\mathbb{R}^n$, Acta Math. Appl. Sinica, pp. 84-91, 1998.

[15] H. L. Royden, Real Analysis, Macmillan Publishing co., New York, 1968

[16] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailiov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter, Berlin and New York, 1995.

[17] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, London, 1944.